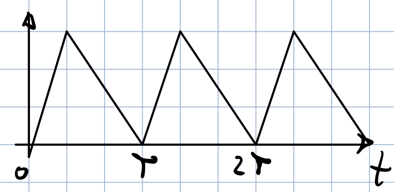
SEGNALI nel DOMINIO delle FREQUENZE

SERIE di FUNZIONI : $S(x) = \sum_{0}^{\infty} f_n(x)$

(es: SERIE di POTENZE : $\sum_{0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} = 2$)

Jean Baptiste FOURIER (1822) :

Data una FUNZIONE PERIODICA, di PERIODO T :



$f : f(t) = f(t+T), \quad \forall t, T \in \mathbb{R}$

se $f(t)$ rispetta opportune condizioni di "REGOLARITÀ" (condizioni di DIRICHLET)

ALLORA :

$f(t)$ è ESPRIMIBILE come SOMMA di SINUSOIDI con FREQUENZE MULTIPLE di $f_1 = \frac{1}{T}$ della FREQUENZA FONDAMENTALE

frequenze : $f_n \begin{cases} n=0 \rightarrow f_0 = 0 \ \text{(COSTANTE)} & \text{COMPONENTE CONTINUA} \\ n=1 \rightarrow f_1 = \frac{1}{T} & \text{FREQUENZA FONDAMENTALE} \\ n>1 \rightarrow f_n = \frac{n}{T} & n\text{-ESIMA ARMONICA} \end{cases}$

SVILUPPO in SERIE di FOURIER

in FORMA ESPONENZIALE : (per $f(t) \in \mathbb{C}$)

SINUSOIDI, FREQ. $f_n = \frac{n}{T}$

$$f(t) = \sum_{-\infty}^{+\infty} c_n \underbrace{e^{j 2\pi \frac{n}{T} t}}_{\text{SINUS. COMPLESSA}} \qquad \left[e^{j2\pi \frac{n}{T} t} = \cos\left(2\pi \frac{n}{T} t\right) + j \sin\left(2\pi \frac{n}{T} t\right) \right]$$

dove : $c_n = \frac{1}{T} \int_T f(t)\, e^{-j 2\pi \frac{n}{T} t}\, dt$

da $\{c_n\} \longrightarrow f(t)$ EQUAZIONE di SINTESI

da $f(t) \longrightarrow \{c_n\}$: EQUAZIONE di ANALISI

in FORMA TRIGONOMETRICA (per $f(t) \in \mathbb{R}$) :

$$f(t) = c_0 + 2 \sum_{1}^{\infty} \rho_n \cos\left(2\pi \frac{n}{T} t + \vartheta_n\right), \quad \text{dove: } \rho_n = |c_n|, \ \vartheta_n = \measuredangle c_n$$

$$= c_0 + 2 \sum_{1}^{\infty} \left[a_n \cos\left(2\pi \frac{n}{T} t\right) + b_n \sin\left(2\pi \frac{n}{T} t\right) \right] \quad \text{dove } a_n = \mathrm{Re}[c_n], \ b_n = -\mathrm{Im}[c_n]$$

Dalla forma ESPONENZIALE alla TRIGONOMETRICA:

Ipotesi: $f(t) \in \mathbb{R}, \forall t \in \mathbb{R}$

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{j2\pi \frac{n}{T} t} = c_0 + \sum_{1}^{\infty} \left[c_n e^{j2\pi \frac{n}{T} t} + c_{-n} e^{-j2\pi \frac{n}{T} t} \right] =$$

considero c_{-n}:

$$c_{-n} = \frac{1}{T} \int_T f(t) e^{j2\pi \frac{n}{T} t} dt = (f(t) \in \mathbb{R}) = \frac{1}{T} \int_T \overline{f(t) e^{-j2\pi \frac{n}{T} t}} dt = \overline{c_n}$$

$$\rightarrow f(t) = c_0 + \sum_{1}^{\infty} \left[c_n e^{j2\pi \frac{n}{T} t} + \overline{c_n} \underbrace{e^{-j2\pi \frac{n}{T} t}}_{\overline{e^{j2\pi \frac{n}{T} t}}} \right] = c_0 + 2 \sum_{1}^{\infty} \mathrm{Re}\left[c_n e^{j2\pi \frac{n}{T} t} \right] =$$

(Esprimo: $c_n = \rho_n e^{j\vartheta_n}$, $\rho_n = |c_n|$, $\vartheta_n = \measuredangle c_n$)

$$f(t) = c_0 + 2 \sum_{1}^{\infty} \mathrm{Re}\left[\rho_n e^{j\vartheta_n} e^{j2\pi \frac{n}{T} t} \right] = c_0 + 2 \sum_{1}^{\infty} \mathrm{Re}\left[\rho_n e^{j(2\pi \frac{n}{T} t + \vartheta_n)} \right] =$$

$$f(t) = c_0 + 2 \sum_{1}^{\infty} \rho_n \cos\left(2\pi \frac{n}{T} t + \vartheta_n\right)$$

dove: $\rho_n = |c_n|$; $\vartheta_n = \measuredangle c_n$

I FORMA TRIGONOMETRICA

Definisco: $c_n = a_n - jb_n \longrightarrow a_n = \mathrm{Re}[c_n]$, $b_n = -\mathrm{Im}[c_n]$

$$\rightarrow f(t) = c_0 + 2 \sum_{1}^{\infty} \mathrm{Re}\left[(a_n - jb_n) e^{j2\pi \frac{n}{T} t} \right] =$$

$$= c_0 + 2 \sum_{1}^{\infty} \mathrm{Re}\left[a_n e^{j2\pi \frac{n}{T} t} - j b_n e^{j2\pi \frac{n}{T} t} \right] =$$

$j = e^{j\frac{\pi}{2}}$

$$= c_0 + 2 \sum_{1}^{\infty} \mathrm{Re}\left[a_n e^{j2\pi \frac{n}{T} t} - b_n e^{j(2\pi \frac{n}{T} t + \frac{\pi}{2})} \right] =$$

$\cos(\alpha + \frac{\pi}{2}) = -\sin\alpha$

$$= c_0 + 2 \sum_{1}^{\infty} \left[a_n \cos\left(2\pi \frac{n}{T} t\right) - b_n \cos\left(2\pi \frac{n}{T} t + \frac{\pi}{2}\right) \right] =$$

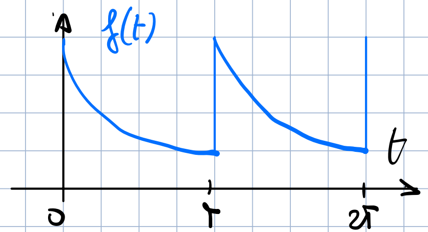
$$f(t) = c_0 + 2 \sum_{1}^{\infty} \left[a_n \cos\left(2\pi \frac{n}{T} t\right) + b_n \sin\left(2\pi \frac{n}{T} t\right) \right]$$

II FORMA TRIGONOMETRICA

$$\text{dove} \begin{cases} a_n = \mathrm{Re}[c_n] = \mathrm{Re}\left[\frac{1}{T} \int_T f(t) e^{-j2\pi \frac{n}{T} t} dt \right] = \frac{1}{T} \int_T f(t) \cos\left(2\pi \frac{n}{T} t\right) dt \\ b_n = -\mathrm{Im}[c_n] = -\mathrm{Im}\left[\frac{1}{T} \int_T f(t) e^{-j2\pi \frac{n}{T} t} dt \right] = \frac{1}{T} \int_T f(t) \sin\left(2\pi \frac{n}{T} t\right) dt \end{cases}$$

EQQ. di ANALISI

$f(t)$,
DOMINIO: TEMPO
$t \in \mathbb{R}$

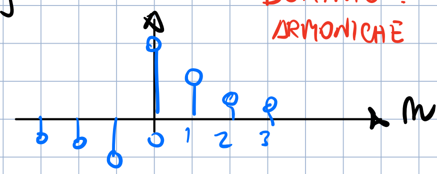


ANALISI $\longrightarrow$

$\longleftarrow$ SINTESI

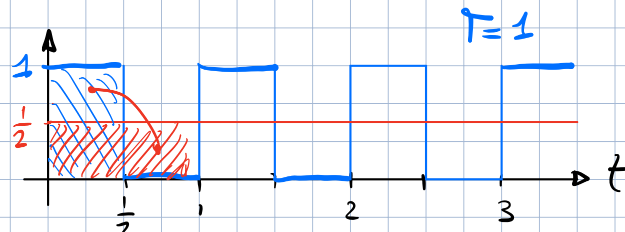
$\{c_n\},\ n \in \mathbb{Z}$
$\{a_n, b_n\}$

$\{c_n\},\ n \in \mathbb{Z}$
DOMINIO: ARMONICHE



ESEMPIO: calcolo SERIE di FOURIER di un'onda QUADRA:

$$s(t) = \begin{cases} 1 & k \le t < k+\frac{1}{2} \\ 0 & k+\frac{1}{2} \le t < k+1 \end{cases}, \quad k \in \mathbb{Z}$$



$n = 0$:

$$c_0 = \frac{1}{T}\int_T s(t)\, e^{-j2\pi\frac{n}{T}t}\, dt = \begin{pmatrix}\text{VALOR MEDIO}\\ \text{di } s(t)\end{pmatrix} = \int_0^1 s(t)\,dt = \int_0^{\frac{1}{2}} 1\, dt = [t]_0^{1/2} = 1/2$$

$n \neq 0$:

$$c_n = \frac{1}{T}\int_T s(t)\, e^{-j2\pi\frac{n}{T}t}\, dt = \int_0^1 s(t)\, e^{-j2\pi\frac{n}{\cancel{T}}t}\, dt = \int_0^{\frac{1}{2}} e^{-j2\pi n t}\, dt =$$

$$= \frac{-1}{j2\pi n}\left[e^{-j2\pi n t}\right]_0^{1/2} = \frac{j}{2\pi n}\left[e^{-j\pi n} - e^0\right] = \frac{j}{2\pi n}\left[(-1)^n - 1\right] = \begin{cases} \frac{-j}{\pi n}, & n \text{ DISPARI} \\ 0, & n \text{ PARI} \end{cases}$$

$$\left(\int e^{at}\,dt = \frac{1}{a}e^{at} + c\right)$$

Quindi:
$$s(t) = \frac{1}{2} + 2\sum_{\substack{n=1 \\ (n \text{ DISPARI})}}^{\infty} \frac{-j}{\pi n} e^{j2\pi n t} = \frac{1}{2} + 2\sum_{n=0}^{\infty} \frac{-j}{\pi(2n+1)} e^{-j2\pi(2n+1)t}$$

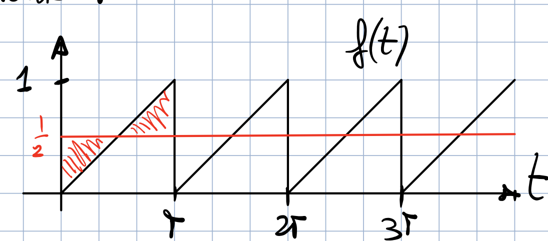
Espresso in II forma TRIGONOMETRICA:

$$a_n = \mathrm{Re}[c_n] = 0,\ \forall n \quad ; \quad b_n = -\mathrm{Im}[c_n] = \begin{cases} 0 & n \text{ PARI} \\ \frac{1}{\pi n} & n \text{ DISPARI} \end{cases}$$

$$\rightarrow s(t) = \frac{1}{2} + 2\sum_{\substack{n=1 \\ n \text{ DISPARI}}}^{\infty} \frac{1}{\pi n}\sin(2\pi n t) = \frac{1}{2} + 2\sum_{n=0}^{\infty}\frac{1}{\pi(2n+1)}\sin(2\pi(2n+1)t)$$

ESEMPIO: FS del "DENTE di SEGA" di periodo T:

$$\begin{cases} f(t) = \frac{t}{T}, & 0 \le t < T \\ f(t+T) = f(t), & \forall t \in \mathbb{R} \end{cases}$$



$$n=0: \quad c_0 = \frac{1}{T}\int_T f(t)\,dt = \frac{1}{T}\int_0^T \frac{t}{T}\,dt = \frac{1}{T^2}\int_0^T t\,dt = \frac{1}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{1}{\cancel{T^2}}\left[\frac{\cancel{T^2}}{2} - \cancel{0}\right] = \frac{1}{2}$$

$$m\neq 0: \quad c_m=\frac{1}{T}\int_T f(t)\, e^{-j2\pi\frac{m}{T}t}\,dt=\frac{1}{T}\int_0^T \frac{t}{T}\, e^{-j2\pi\frac{m}{T}t}\,dt=\frac{1}{T^2}\int_0^T t\cdot e^{-j2\pi\frac{m}{T}t}\,dt=$$

$g'(t) \rightarrow g(t)=\frac{-T}{j2\pi m}e^{-j2\pi\frac{m}{T}t}$

$f(t)\rightarrow f'(t)=1$

[Integraz. per PARTI: $\int f(t)\cdot g'(t)\,dt=\left[f(t)\cdot g(t)\right]-\int f'(t)g(t)\,dt$]

$$c_n=\frac{1}{T^2}\left\{\left[t\cdot\frac{-T}{j2\pi m}e^{-j2\pi\frac{m}{T}t}\right]_0^T-\int_0^T \cancel{1}\cdot\frac{-T}{j2\pi m}e^{-j2\pi\frac{m}{T}t}\,dt\right\}=$$

$$=\frac{1}{T^2}\left\{\left[\frac{-T^2}{j2\pi m}\underbrace{e^{-j2\pi m}}_{1}-\cancel{0}\right]-\left(\frac{-T}{j2\pi m}\right)^2\left[e^{-j2\pi\frac{m}{T}t}\right]_0^T\right\}=$$

$$=\frac{1}{\cancel{T^2}}\left\{\frac{j\cancel{T^2}}{2\pi m}-\cancel{\left(\frac{-T}{j2\pi m}\right)^2\left[\underbrace{e^{-j2\pi m}}_{1}-\underbrace{e^{0}}_{1}\right]}\right\}=\frac{j}{2\pi m},\quad m\neq 0$$

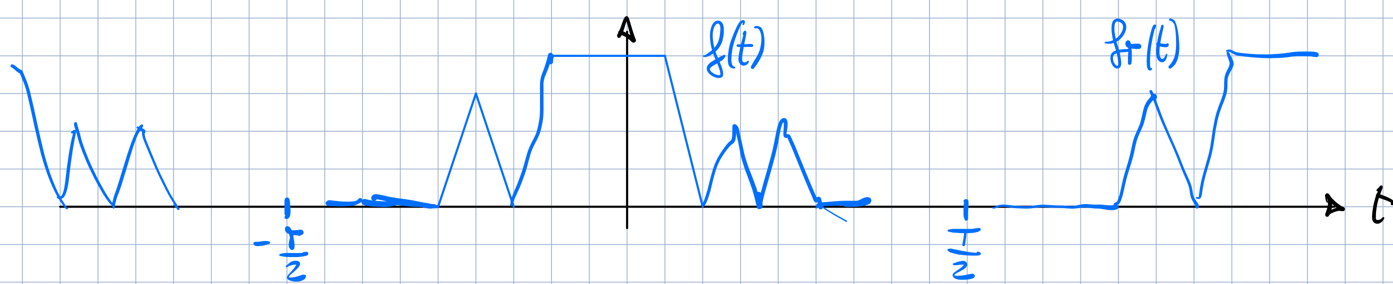
$$\boxed{\begin{array}{l} c_0=\dfrac{1}{2} \\ c_n=\dfrac{j}{2\pi n},\quad n\neq 0\end{array}}$$

Quindi: $f(t)=\frac{1}{2}+\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{j}{2\pi m}e^{j2\pi\frac{m}{T}t}$ forma ESPONENZIALE

$\longrightarrow \quad a_n=\mathrm{Re}[c_n]=0\ ;\quad b_n=-\mathrm{Im}[c_n]=\frac{-1}{2\pi m}$ (II forma TRIGONOMETRICA)

$$f(t)=\frac{1}{2}+\cancel{2}\sum_{m=1}^{\infty}\frac{-1}{\cancel{2}\pi m}\sin\left(2\pi\frac{m}{T}t\right)=\frac{1}{2}-\sum_{m=1}^{\infty}\frac{1}{\pi m}\sin\left(2\pi\frac{m}{T}t\right)$$

Possiamo GENERALIZZARE lo SV. in SERIE di FOURIER per QUALUNQUE SEGNALE? (cioè per SEGNALI NON PERIODICI) ?



Dato $f(t)$ NON PERIODICO,

$$\text{def}: \quad f_T(t)=\begin{cases} f(t) & \text{per } -\frac{T}{2}\leq t<\frac{T}{2}\\ f(t-kT), & k\in\mathbb{Z} \text{ altrove}\end{cases}\ \longrightarrow\ f_T(t) \text{ è PERIODICO con PERIODO} = T$$

Posso scrivere: $f(t)=\lim_{T\to\infty} f_T(t)$ $\quad f(t)$ PERIODICO, $T\to\infty \Rightarrow$ NON PERIODICO!

Dato che $f_T(t)$ è PERIODICO $\longrightarrow$ posso applicare SF :

$$f_T(t) = \sum_{-\infty}^{\infty}{}_n \, c_n e^{j2\pi \frac{n}{T} t} = \left\{ def: \frac{n}{T} = f_n \right\} = \sum_{-\infty}^{\infty}{}_n \, c_n e^{j2\pi f_n t}$$

dove : $$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} f_T(t) e^{-j2\pi f_n t} \, dt = \left\{ def: \Delta f = \frac{1}{T} \right\} = \Delta f \underbrace{\int_{-\frac{T}{2}}^{+\frac{T}{2}} f_T(t) e^{-j2\pi f_n t} \, dt}_{c_n}$$

Quindi: $$f_T(t) = \sum_{-\infty}^{\infty}{}_n \left[\Delta f \int_{-\frac{T}{2}}^{+\frac{T}{2}} f_T(\tau) e^{-j2\pi f_n \tau} d\tau \right] e^{j2\pi f_n t}$$

Allora : $$f(t) = \lim_{T \to \infty} f_T(t) = \lim_{T \to \infty} \sum_{-\infty}^{\infty}{}_n \underbrace{\int_{-\frac{T}{2}}^{+\frac{T}{2}} \cancel{f_T}(\tau) e^{-j2\pi f_n \tau} d\tau}_{\xrightarrow{\lim_{T\to\infty}} F(f_n)} \cdot e^{j2\pi f_n t} \cdot \Delta f$$

Definisco : $$F(f) = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi f t} dt$$

Quindi : $$f(t) = \lim_{\substack{T \to \infty \\ \Delta f \to 0}} \sum_{-\infty}^{\infty}{}_n F(f_n) e^{j2\pi f_n t} \cdot \Delta f = \int_{-\infty}^{+\infty} F(f) e^{j2\pi f t} df$$

IN SINTESI :

$$F(f) = \int_{-\infty}^{+\infty} f(t) e^{-j2\pi f t} dt$$ TRASFORMATA di FOURIER di $f(t)$ (eq. di ANALISI)

SPETTRO di $f(t)$

$$F(f) = \mathcal{F}\{f(t)\} \quad ; \quad f(t) \xrightarrow{\mathcal{F}} F(f)$$

$$f(t) = \int_{-\infty}^{+\infty} F(f) e^{j2\pi f t} df$$ (TRASFORMATA INVERSA) ANTITRASFORMATA di FOURIER (eq. di SINTESI)

$$f(t) = \mathcal{F}^{-1}\{F(f)\} \quad ; \quad F(f) \xrightarrow{\mathcal{F}^{-1}} f(t)$$

$F(f)$ è lo SPETTRO di $f(t)$: è una FUNZIONE COMPLESSA di f

$$F(f) : \mathbb{R} \longrightarrow \mathbb{C} \qquad F(f) \in \mathbb{C}$$

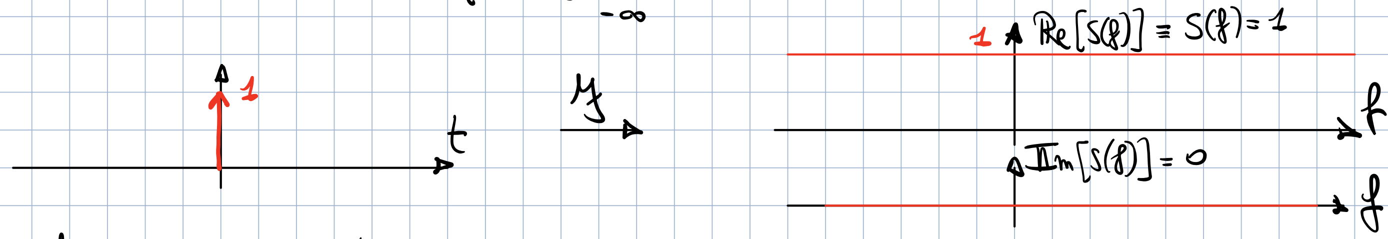
$|F(f)|$ è lo SPETTRO di AMPIEZZA di $f(t)$

$\measuredangle F(f)$ è lo SPETTRO di FASE di $f(t)$

ESEMPI NOTEVOLI di TRASFORMATA

DELTA di DIRAC : $\delta(t)$

$$s(t) = \delta(t) \xrightarrow{\mathcal{F}} S(f) = \mathcal{F}\{s(t)\} = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j2\pi f t}\, dt = e^{-j2\pi f \cdot 0} = 1$$



In generale, per $\delta()$ in t_0 :

$$s(t) = \delta(t-t_0) \xrightarrow{\mathcal{F}} S(f) = \int_{-\infty}^{+\infty} \delta(t-t_0)\, e^{-j2\pi f t}\, dt = e^{-j2\pi f t_0}$$

FOURIER PAIRS :

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$

$$\delta(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j2\pi f t_0} = \cos(2\pi t_0 f) - j\sin(2\pi t_0 f)$$

RETTANGOLO : $\mathrm{rect}(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{altrove} \end{cases}$

$$s(t) = \mathrm{rect}(t) \longrightarrow S(f) = \int_{-\infty}^{+\infty} \mathrm{rect}(t)\, e^{-j2\pi f t}\, dt = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cancel{1} \cdot e^{-j2\pi f t}\, dt = \left[\frac{-1}{j2\pi f}\, e^{-j2\pi f t}\right]_{-\frac{1}{2}}^{+\frac{1}{2}} =$$

$$= \frac{-1}{j2\pi f}\left[e^{-j\pi f} - e^{+j\pi f}\right] = \qquad (\text{Eulero} : e^{j\vartheta} = \cos\vartheta + j\sin\vartheta)$$

$$= \frac{-1}{j2\pi f}\left[\cancel{\cos(\pi f)} - j\sin(\pi f) - \cancel{\cos(\pi f)} - j\sin(\pi f)\right] =$$

$$= \frac{\cancel{-}1}{\cancel{j2}\pi f}\left[\cancel{-2j}\sin(\pi f)\right] = \frac{\sin(\pi f)}{\pi f} = \mathrm{sinc}(f) \; : \text{SENO CARDINALE}$$

$$\mathrm{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\sin \pi f}{\pi f} = \mathrm{sinc}(f) \qquad \text{FOURIER PAIR}$$

TRASFORMATA di FOURIER di SEGNALI REALI

Ipotesi : $s(t) \in \mathbb{R} \quad \forall t \in \mathbb{R}$

$$s(t) \xrightarrow{\mathcal{F}} S(f) = \int_{-\infty}^{+\infty} s(t)\, \underbrace{e^{-j2\pi f t}}_{\text{EULERO}}\, dt = \int_{-\infty}^{+\infty} s(t)\left[\cos(2\pi f t) - j\sin(2\pi f t)\right] dt =$$

$$S(f) = \underbrace{\int_{-\infty}^{\infty} s(t)\cos(2\pi f t)\, dt}_{\in \mathbb{R}} - j \underbrace{\int_{-\infty}^{\infty} s(t)\sin(2\pi f t)\, dt}_{\in \mathbb{R}} = \mathrm{Re}[S(f)] - j\,\mathrm{Im}[S(f)]$$

$$\mathrm{Re}[S(f)] = \int_{-\infty}^{\infty} s(t)\cos(2\pi f t)\, dt = \int_{-\infty}^{\infty} s(t)\cos(2\pi(-f)t)\, dt = \mathrm{Re}[S(-f)]$$

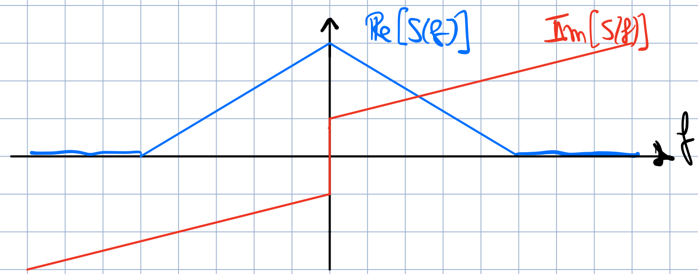
$\cos(\alpha) = \cos(-\alpha)$

$\mathrm{Re}[S(f)]$ ha SIMMETRIA PARI : $\mathrm{Re}[S(f)] = \mathrm{Re}[S(-f)]$

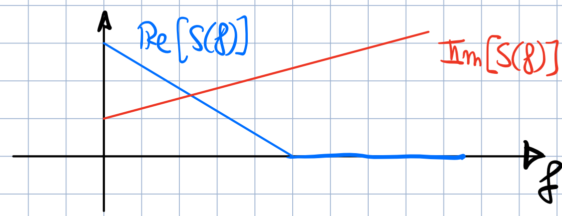
$$\mathrm{Im}[S(f)] = \int_{-\infty}^{\infty} s(t)\sin(2\pi f t)\, dt = -\int_{-\infty}^{\infty} s(t)\sin(2\pi(-f)t)\, dt = -\mathrm{Im}[S(-f)]$$

$\sin(\alpha) = -\sin(-\alpha)$

$\mathrm{Im}[S(f)]$ ha SIMMETRIA DISPARI : $\mathrm{Im}[S(f)] = -\mathrm{Im}[S(-f)]$

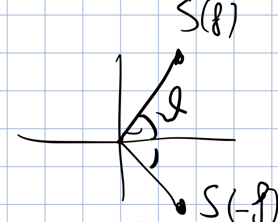


TRASF. di FOURIER BILATERA



TRASF. di FOURIER MONOLATERA

Consideriamo MODULO e FASE :



MODULO : $|S(f)| = |S(-f)|$ ha SIMMETRIA PARI

FASE : $\measuredangle S(f) = -\measuredangle S(-f)$ ha SIMMETRIA DISPARI

Se $s(t)$ è REALE : $s(t) \in \mathbb{R}, \forall t \longleftrightarrow S(f) = \overline{S(-f)}, \forall f$

SIMMETRIA HERMITIANA

BANDA di un SEGNALE

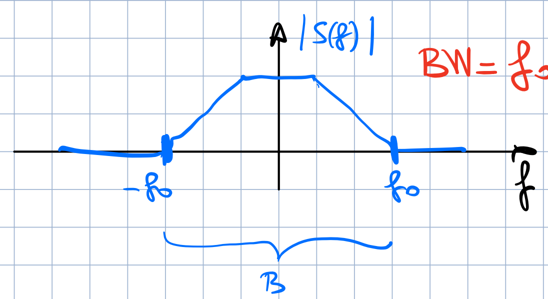
BANDA : il SUPPORTO dello SPETTRO del SEGNALE :

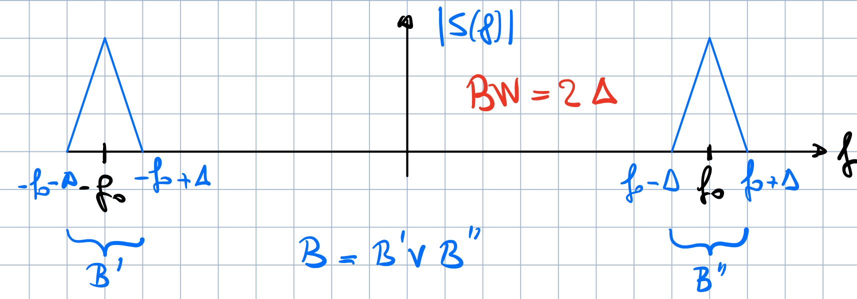
Dato $s(t) \xrightarrow{\mathcal{F}} S(f)$: BANDA di $s(t)$ $:= B = \{ f \in \mathbb{R} : S(f) \neq 0 \}$

LARGHEZZA di BANDA (BANDWIDTH) : ESTENSIONE della BANDA B

★ quando B CONTIENE $f=0$ $\longrightarrow$ $s(t)$ è un SEGNALE in BANDA BASE

★ quando B NON CONTIENE $f=0$ $\longrightarrow$ $s(t)$ è un SEGNALE in BANDA PASSANTE





- La LARGHEZZA di BANDA (BW) di un segnale viene definita sulla RAPPRESENTAZIONE MONOLATERA del suo SPETTRO.

PROPRIETÀ della TRASFORMATA di FOURIER

LINEARITÀ

Se $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$, $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(f)$ $\longrightarrow$ $a x(t) + b y(t) \overset{\mathcal{F}}{\longleftrightarrow} a X(f) + b Y(f)$

DIM: $$\mathcal{F}\{a x(t) + b y(t)\} = \int_{-\infty}^{+\infty} (a x(t) + b y(t)) e^{-j2\pi f t} dt = \int_{-\infty}^{+\infty} a x(t) e^{-j2\pi f t} dt + \int_{-\infty}^{\infty} b y(t) e^{-j2\pi f t} dt =$$

$$= a \underbrace{\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt}_{X(f)} + b \underbrace{\int_{-\infty}^{\infty} y(t) e^{-j2\pi f t} dt}_{Y(f)} = a X(f) + b Y(f) \rightarrow \text{è LINEARE.}$$

DUALITÀ (o SIMMETRIA)

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) \longleftrightarrow S(t) \overset{\mathcal{F}}{\longleftrightarrow} s(-f)$$

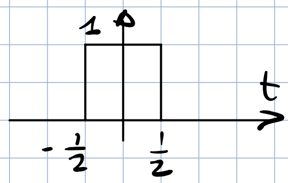
DIM: $$\mathcal{F}\{S(t)\} = \int_{-\infty}^{+\infty} S(t) e^{-j2\pi f t} dt = \underbrace{\int_{-\infty}^{\infty} S(t) e^{j2\pi(-f) t} dt} = s(-f) \quad \text{CVD}$$

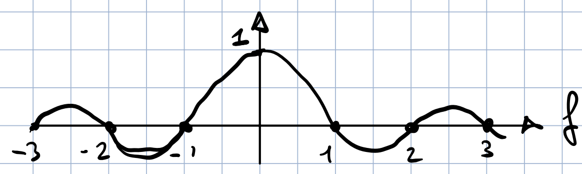
Se $s(t)$ ha SIMMETRIA PARI: $s(t) = s(-t)$, $\forall t$

Se $s(t) = s(-t)$, $\forall t$ $\longrightarrow$ $S(t) \overset{\mathcal{F}}{\longleftrightarrow} s(-t) = s(t)$

↳ $s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) \longleftrightarrow S(t) \overset{\mathcal{F}}{\longleftrightarrow} s(f)$

ESEMPIO : $s(t) = rect(t) \longrightarrow S(f) = sinc(f)$





$\rightarrow$ per DUALITÀ : $s(t) = sinc(t) \longrightarrow S(f) = rect(-f) = rect(f)$

TRASLAZIONE nei TEMPI

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) \longrightarrow s(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} S(f)e^{-j2\pi f t_0}$$

DIM: $\mathcal{F}\{s(t-t_0)\} = \int_{-\infty}^{\infty} s(t-t_0)\, e^{-j2\pi f t}\, dt = \left\{ \begin{array}{l} t' = t - t_0 \rightarrow t = t' + t_0 \\ dt' = dt \end{array} \right\} =$

$$= \int_{-\infty}^{\infty} s(t')\, e^{-j2\pi f(t'+t_0)}\, dt' = \int_{-\infty}^{\infty} s(t')\, e^{-j2\pi f t'} \cdot \underbrace{e^{-j2\pi f t_0}}_{\text{COSTANTE}}\, dt' =$$

$$= e^{-j2\pi f t_0} \underbrace{\int_{-\infty}^{+\infty} s(t')\, e^{-j2\pi f t'}\, dt'}_{S(f)} = S(f)\, e^{-j2\pi f t_0} \qquad \text{CVD}$$

TRASLAZIONE nelle FREQUENZE (x DUALITÀ)

Pr. trasl. TEMPI : $s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) \longrightarrow s(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} S(f)\, e^{-j2\pi f t_0}$

$\rightarrow$ per DUALITÀ : " $\longrightarrow s(t)\, e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} S(f+f_0)$

SPETTRO del SEGNALE MODULATO

MODULAZIONE di $s(t)$: $s_M(t) = s(t)\cos(2\pi f_c t)$ $\rightarrow$ freq. PORTANTE

$$s_M(t) = s(t)\, \frac{1}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right] =$$

$$= \frac{1}{2}\left[s(t)\, e^{j2\pi f_c t} + s(t)\, e^{-j2\pi f_c t}\right]$$

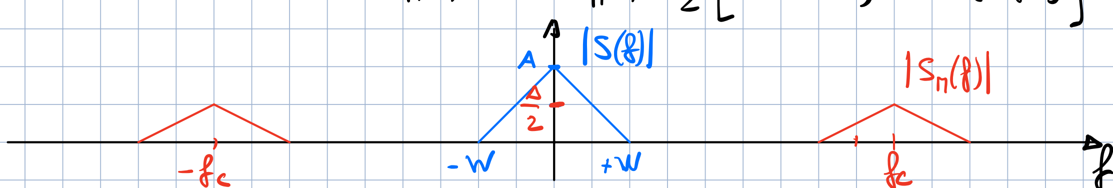
EULERO :

$e^{j\vartheta} = \cos\vartheta + j\sin\vartheta$

$e^{-j\vartheta} = \cos\vartheta - j\sin\vartheta$

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) \rightarrow s_M(t) \overset{\mathcal{F}}{\longleftrightarrow} S_M(f) = \frac{1}{2}\left[S(f-f_c) + S(f+f_c)\right]$$

$e^{j\vartheta} + e^{-j\vartheta} = 2\cos\vartheta$



$\cos\vartheta = \frac{1}{2}\left[e^{j\vartheta} + e^{-j\vartheta}\right]$

$\sin\vartheta = \frac{1}{2j}\left[e^{j\vartheta} - e^{-j\vartheta}\right]$

SCALATURA

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f) \longrightarrow s(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} S\left(\frac{f}{a}\right)$$

DIM: 1) $a>0$: $\mathcal{F}\{s(at)\} = \int_{-\infty}^{+\infty} s(at)\, e^{-j2\pi f t}\, dt = \begin{Bmatrix} \tau = at \\ d\tau = a\, dt \end{Bmatrix} = \int_{-\infty}^{+\infty} s(\tau)\, e^{-j2\pi f \frac{\tau}{a}} \cdot \frac{d\tau}{a} =$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} s(\tau)\, e^{-j2\pi \frac{f}{a}\tau} \cdot d\tau = \frac{1}{a} S\left(\frac{f}{a}\right) \checkmark$$

2) $a<0$: $\mathcal{F}\{s(at)\} = \int_{-\infty}^{+\infty} s(at)\, e^{-j2\pi f t}\, dt = \begin{Bmatrix} \tau = at \\ d\tau = a\, dt \end{Bmatrix} = \int_{+\infty}^{-\infty} s(\tau)\, e^{-j2\pi f \frac{\tau}{a}} \frac{d\tau}{a} =$

$$= -\int_{-\infty}^{+\infty} s(\tau)\, e^{-j2\pi \frac{f}{a}\tau} \frac{d\tau}{a} = \frac{1}{-a} \int_{-\infty}^{+\infty} s(\tau)\, e^{-j2\pi \frac{f}{a}\tau}\, d\tau = \frac{1}{-a} S\left(\frac{f}{a}\right)$$

Quindi : $$s(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} S\left(\frac{f}{a}\right)$$

CONVOLUZIONE

$$\begin{matrix} x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) \\ y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(f) \end{matrix} \longrightarrow x(t) * y(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) \cdot Y(f)$$

DIM: $\mathcal{F}\{x(t) * y(t)\} = \mathcal{F}\left\{ \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \right\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \right] e^{-j2\pi f t}\, dt =$ $\quad t \rightarrow (t-\tau)+\tau$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, e^{-j2\pi f (t-\tau)}\, e^{-j2\pi f \tau}\, dt\, d\tau =$$

$$= \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f \tau}\, d\tau}_{\text{DIPENDE SOLO da } \tau \rightarrow X(f)} \cdot y(t-\tau)\, e^{-j2\pi f (t-\tau)}\, dt =$$

$$= \int_{-\infty}^{\infty} X(f)\, y(t-\tau)\, e^{-j2\pi f (t-\tau)}\, dt = X(f) \int_{-\infty}^{+\infty} y(t-\tau)\, e^{-j2\pi f (t-\tau)}\, dt = \begin{Bmatrix} t' = t-\tau \\ dt' = dt \end{Bmatrix} =$$

$$= X(f) \int_{-\infty}^{\infty} \underbrace{y(t')\, e^{-j2\pi f t'}\, dt'}_{Y(f)} = X(f) \cdot Y(f) \quad \text{CVD}$$

$$x(t) * y(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) \cdot Y(f)$$

per DUALITÀ $$x(t) \cdot y(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) * Y(f)$$

DERIVAZIONE

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \longrightarrow s'(t) = \frac{d}{dt}s(t) \xleftrightarrow{\mathcal{F}} j2\pi f \cdot S(f)$$

DIM: $$s'(t) = \frac{d}{dt}s(t) = \frac{d}{dt}\int_{-\infty}^{\infty} S(f)e^{j2\pi ft}\,df = \int_{-\infty}^{+\infty} \underbrace{S(f)\,j2\pi f}_{\mathcal{F}\{s'(t)\}}\, e^{j2\pi ft}\,df = s'(t)$$

INTEGRAZIONE

$$s(t) \xleftrightarrow{\mathcal{F}} S(f) \longrightarrow \int_{-\infty}^{t} s(\tau)\,d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j2\pi f}S(f) + \frac{S(0)}{2}\delta(f)$$

ENERGIA e POTENZA di SEGNALI nel DOMINIO delle FREQUENZE

POTENZA ISTANTANEA di $s(t)$: $P(t) = s(t)\cdot\overline{s(t)} = |s(t)|^2$ $(= s^2(t)$ se $s(t)\in\mathbb{R})$

ENERGIA di $s(t)$: $E_s = \int_{-\infty}^{+\infty}|s(t)|^2dt$

POTENZA MEDIA di $s(t)$: $P_s = \lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{+\infty}|s(t)|^2dt$

$s(t)$ è SEGNALE ENERGIA sse : $0 < E_s < \infty$ (energia FINITA)
$\rightarrow P_s = 0$ (pot. media NULLA)

$s(t)$ è SEGNALE POTENZA sse : $0 < P_s < \infty$ (pot. media FINITA)
$\rightarrow E_s = \infty$ (energia INFINITA)

TEOREMA di PARSEVAL (per segnali ENERGIA)

Sia $s(t)$ un SEGNALE ENERGIA e sia $S(f)$ il SUO SPETTRO

$$\boxed{E_s = \int_{-\infty}^{+\infty}|s(t)|^2dt = \int_{-\infty}^{+\infty}|S(f)|^2df}$$

"L'ENERGIA di $s(t)$ COINCIDE con l'ENERGIA di $S(f)$"

DIM :

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2\, dt = \int_{-\infty}^{\infty} s(t)\cdot\overline{s(t)}\, dt = \int_{-\infty}^{\infty} \overline{s(t)} \underbrace{\int_{-\infty}^{\infty} S(f)\, e^{j2\pi f t}\, df}_{s(t)} \cdot dt =$$

$$= \int_{-\infty}^{\infty} S(f)\left[\int_{-\infty}^{\infty}\overline{s(t)}\, e^{j2\pi f t}\, dt\right] df = \left\{\overline{a\cdot b} = \overline{a}\cdot\overline{b}\right\} = \int_{-\infty}^{\infty} S(f) \underbrace{\overline{\int_{-\infty}^{+\infty} s(t)\, e^{-j2\pi f t}\, dt}}_{\overline{S(f)}} \cdot df =$$

$$= \int_{-\infty}^{+\infty} S(f)\cdot \overline{S(f)}\cdot df = \int_{-\infty}^{+\infty}|S(f)|^2\, df \quad \text{CVD}$$

DEF: $|S(f)|^2$: DENSITÀ SPETTRALE di ENERGIA

TEOREMA di PARSEVAL per SEGNALI POTENZA

Dato $s(t)$ SEGNALE POTENZA, la POTENZA MEDIA P_s:

$$P_s = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}|s(t)|^2\, dt = \int_{-\infty}^{+\infty} \underbrace{|S_p(f)|^2}_{\text{DENSITÀ SPETTRALE di POTENZA}}\, df$$

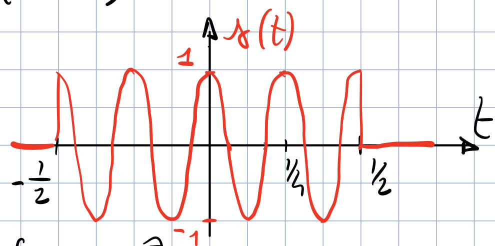
con: $S_p(f) = \lim_{T\to\infty} \frac{|S_T(f)|^2}{T}$, dove: $S_T(f) = \mathcal{F}\left\{s(t)\,\mathrm{rect}\left(\frac{t}{T}\right)\right\}$

ESEMPI di CALCOLO di TRASFORMATA di FOURIER

- calcolare la TR. di FOURIER di: $s(t) = \mathrm{rect}(t)\cos(8\pi t)$

$$s(t) = \mathrm{rect}(t)\cos(8\pi t) = \begin{cases} \cos(8\pi t) & -\frac{1}{2}\le t\le \frac{1}{2} \\ 0 & \text{altrove} \end{cases}$$

$f = 4 \rightarrow T = \frac{1}{4}$



$$s(t) = \underset{\downarrow\mathcal{F}\ \mathrm{sinc}(f)}{\mathrm{rect}(t)} \cdot \underset{\downarrow\mathcal{F}\ \frac{1}{2}[\delta(f+4)+\delta(f-4)]}{\cos(8\pi t)} \longrightarrow S(f) = \mathcal{F}\{\mathrm{rect}(t)\} * \mathcal{F}\{\cos(8\pi t)\}$$

$$\cos(8\pi t) = \frac{1}{2}\left[e^{j8\pi t} + e^{-j8\pi t}\right] \xrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta(f-4)+\delta(f+4)\right]$$

$$S(f) = \mathrm{sinc}(f) * \frac{1}{2}\left[\delta(f-4)+\delta(f+4)\right] = \frac{1}{2}\mathrm{sinc}(f)*\delta(f-4) + \frac{1}{2}\mathrm{sinc}(f)*\delta(f+4) =$$

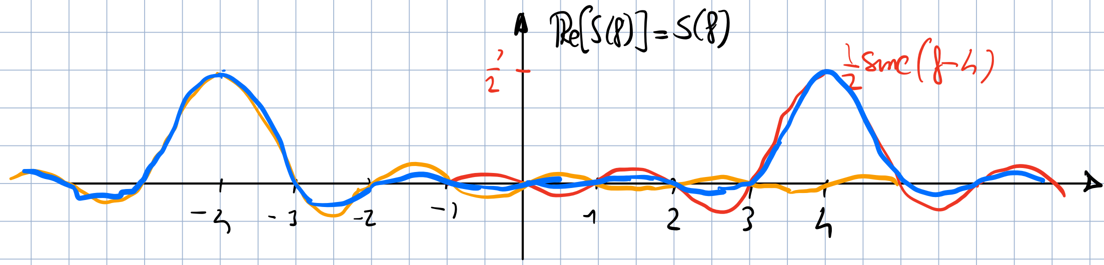
$$= \frac{1}{2}\left[\mathrm{sinc}(f-4) + \mathrm{sinc}(f+4)\right] \quad ✓$$

Applicando le prop. della MODULAZIONE :

$$s(t)\cos(2\pi f_c t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[S(f-f_c)+S(f+f_c)\right]$$

$$s(t)=\mathrm{rect}(t) \longrightarrow S(f)=\mathrm{sinc}(f)$$

$$\mathrm{rect}(t)\cos(8\pi t) \longrightarrow S(f)=\frac{1}{2}\left[\mathrm{sinc}(f-4)+\mathrm{sinc}(f+4)\right]$$



• Calcolare la FT di : $s(t)=\mathrm{sinc}\left(\frac{t-4}{2}\right)$ e rappresentare: Re, Im, | |, ∡

Tecnica: parto da FOURIER PAIR e applico PROPRIETÀ della FT :

$$s'(t)=\mathrm{sinc}(t) \xrightarrow{\mathcal{F}} S'(f)=\mathrm{rect}(f)$$

SCALATURA : $$s''(t)=\mathrm{sinc}\left(\frac{t}{2}\right) \xrightarrow{\mathcal{F}} S''(f)=2\,\mathrm{rect}(2f)$$

TRASLAZIONE : $$s(t)=\mathrm{sinc}\left(\frac{t-4}{2}\right) \xrightarrow{\mathcal{F}} S(f)=2\,\mathrm{rect}(2f)\,e^{-j2\pi f4}$$

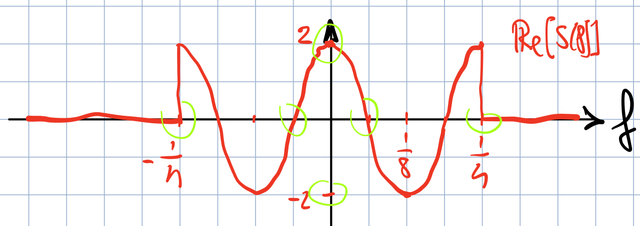
$$S(f)=2\,\mathrm{rect}(2f)\,e^{-j8\pi f}$$

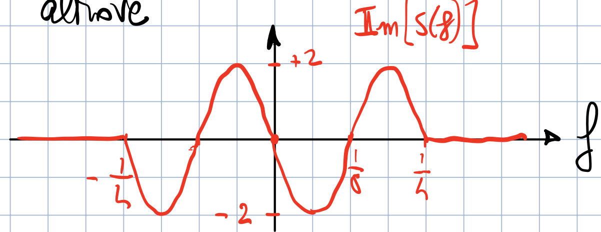
Troviamo $\mathrm{Re}[S(f)]$ e $\mathrm{Im}[S(f)]$:

$$S(f)=2\,\mathrm{rect}(2f)\left[\cos(-8\pi f)+j\sin(-8\pi f)\right]=\underbrace{2\,\mathrm{rect}(2f)\cos(8\pi f)}_{\mathrm{Re}[S(f)]}-j\underbrace{2\,\mathrm{rect}(2f)\sin(8\pi f)}_{\mathrm{Im}[S(f)]}$$

$$\rightarrow \mathrm{Re}[S(f)]=2\,\mathrm{rect}(2f)\cos(8\pi f)=\begin{cases}2\cos(8\pi f) & -\frac{1}{4}\le f\le\frac{1}{4}\\ 0 & \text{altrove}\end{cases}$$

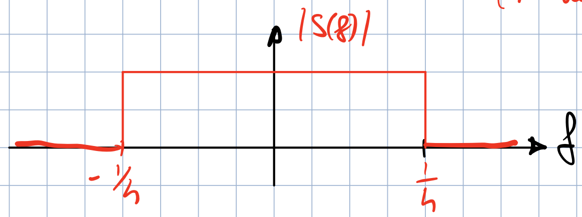
$$\rightarrow \mathrm{Im}[S(f)]=-2\,\mathrm{rect}(2f)\sin(8\pi f)=\begin{cases}-2\sin(8\pi f) & -\frac{1}{4}\le f\le\frac{1}{4}\\ 0 & \text{altrove}\end{cases}$$

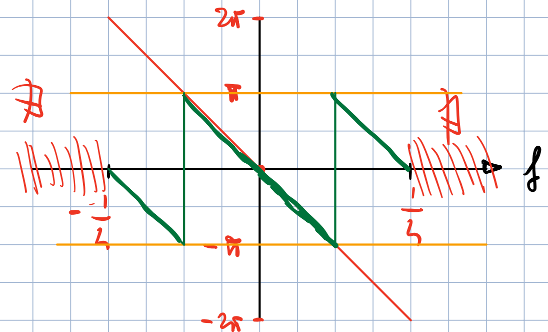




$$|S(f)| = |2\,\mathrm{rect}(2f)\,e^{-j8\pi f}| = |2|\cdot|\mathrm{rect}(2f)|\,\underbrace{|e^{-j8\pi f}|}_{=1} = 2\,\mathrm{rect}(2f)\cdot \cancel{1} = \begin{cases} 2, & -\frac{1}{4} \le f \le \frac{1}{4} \\ 0 & \text{altrove} \end{cases}$$

$$\measuredangle S(f) = \measuredangle\{2\,\mathrm{rect}(2f)\,e^{-j8\pi f}\} = \underbrace{\measuredangle[2\,\mathrm{rect}(2f)]}_{\begin{cases} 0, & -\frac{1}{4} \le f \le \frac{1}{4} \\ \nexists & \text{altrove} \end{cases}} + \underbrace{\measuredangle[e^{-j8\pi f}]}_{-8\pi f} = \begin{cases} -8\pi f, & -\frac{1}{4} \le f \le \frac{1}{4} \\ \nexists & \text{altrove} \end{cases}$$





- Calcolare la FT di: $s(t) = 8\,\mathrm{sinc}^2\left(\frac{t-3}{2}\right)$ e rappresentare $|S(f)|$, $\measuredangle S(f)$

$$s(t) = 8\,\mathrm{sinc}\left(\frac{t-3}{2}\right)\cdot \mathrm{sinc}\left(\frac{t-3}{2}\right)$$

$$\mathrm{sinc}(t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}(f)$$

$$\mathrm{sinc}^2(t) = \mathrm{sinc}(t)\cdot\mathrm{sinc}(t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}(f) * \mathrm{rect}(f) = \mathrm{tri}(f)$$

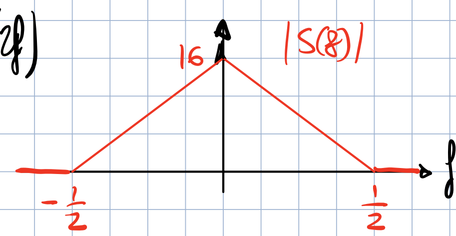
scalatura: $\mathrm{sinc}^2\left(\frac{t}{2}\right) \xleftrightarrow{\mathcal{F}} 2\,\mathrm{tri}(2f)$

traslazione: $\mathrm{sinc}^2\left(\frac{t-3}{2}\right) \xleftrightarrow{\mathcal{F}} 2\,\mathrm{tri}(2f)\,e^{-j6\pi f}$

$$S(f) = 16\,\mathrm{tri}(2f)\,e^{-j6\pi f}$$

MODULO:

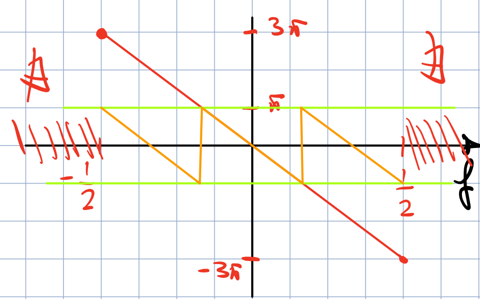
$$|S(f)| = |16\,\mathrm{tri}(2f)\,e^{-j6\pi f}| = |16\,\mathrm{tri}(2f)|\cdot\underbrace{|e^{-j6\pi f}|}_{=1} = 16\,\mathrm{tri}(2f)$$



FASE:

$$\measuredangle S(f) = \measuredangle\{16\,\mathrm{tri}(2f)\,e^{-j6\pi f}\} = \underbrace{\measuredangle[16\,\mathrm{tri}(2f)]}_{\begin{cases} 0, & -\frac{1}{2} \le f \le \frac{1}{2} \\ \nexists & \text{altrove} \end{cases}} + \underbrace{\measuredangle[e^{-j6\pi f}]}_{-6\pi f} =$$

$$= \begin{cases} -6\pi f, & -\frac{1}{2} \le f \le \frac{1}{2} \\ \nexists & \text{altrove} \end{cases}$$



SISTEMI LTI nel DOMINIO della FREQUENZA

Considero un generico sistema LTI :

$$\begin{tikzcd} x(t) \arrow[r] \arrow[d, "\mathcal{F}"'] & \boxed{\text{SISTEMA LTI } h(t)} \arrow[r] \arrow[d, "\mathcal{F}"] & \boxed{y(t) = x(t) * h(t)} \arrow[d, "\mathcal{F}"'] \\ X(f) & H(f) & \boxed{Y(f) = X(f) \cdot H(f)} \end{tikzcd}$$

Nel DOMINIO delle FREQUENZE:

RISPOSTA in FREQUENZA

RISPOSTA in FREQUENZA $H(f)$: TRASF. di FOURIER della RISPOSTA all'IMPULSO $h(t)$

$$h(t) \xleftrightarrow{\mathcal{F}} H(f) = \int_{-\infty}^{+\infty} h(t)\, e^{-j2\pi f t}\, dt$$

$$\begin{tikzcd} x(t) \arrow[r] \arrow[d] & \boxed{\text{LTI } h(t)} \arrow[r] \arrow[d] & y(t) = x(t) * h(t) \arrow[d, phantom] \\ X(f) & H(f) & Y(f) = X(f) \cdot H(f) \end{tikzcd}$$

CONVOLUZIONE

PRODOTTO

Nuovo strumento di analisi dei SISTEMI LTI

Esempio : ANALISI in FREQUENZA dei SISTEMI LTI :

$$\begin{tikzcd} \text{dato } x(t) \arrow[r, "*"', "h(t)"'] \arrow[d, "\mathcal{F}"'] & y(t) = x(t) * h(t) \\ X(f) \arrow[r, "H(f)"'] & Y(f) = X(f)\, H(f) \arrow[u, "\mathcal{F}^{-1}"'] \end{tikzcd}$$

SIGNIFICATO della RISPOSTA in FREQUENZA

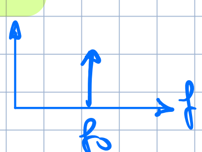
Consideriamo la RISPOSTA di $S[\cdot]$ a una SINUSOIDE COMPLESSA a frequenza f_0 :

$$\begin{tikzcd} x_{f_0}(t) \arrow[r] & \boxed{S[\cdot] \text{ LTI } \; h(t) / H(f)} \arrow[r] & y_{f_0}(t) \end{tikzcd}$$

$$x_{f_0}(t) = e^{j2\pi f_0 t}$$

$\downarrow \mathcal{F}$

$$X_{f_0}(f) = \delta(f - f_0)$$



$$\longrightarrow Y_{f_0}(f) = X_{f_0}(f) \cdot H(f) = \delta(f - f_0)\, H(f) = H(f_0)\, \delta(f - f_0)$$

$\mathcal{F}^{-1} \downarrow$

L'USCITA $y_{f_0}(t)$ è ANCORA una SINUSOIDE alla STESSA FREQUENZA f_0! ← $y_{f_0}(t) = H(f_0)\, e^{j2\pi f_0 t}$

MODULO: $|y_{f_0}(t)| = |H(f_0)| \cdot |x_{f_0}(t)|$

FASE: $\measuredangle y_{f_0}(t) = \measuredangle H(f_0) + \measuredangle x_{f_0}(t)$

$\rightarrow \begin{cases} |H(f)| : \text{RISPOSTA in AMPIEZZA} \\ \measuredangle H(f) : \text{RISPOSTA in FASE} \end{cases}$

- La componente SINUSOIDALE a frequenza f dell'INGRESSO con SPETTRO $X(f)$ viene MOLTIPLICATA per $H(f)$ a dare la comp. a frequenza f dello SPETTRO dell'USCITA $Y(f)$: $Y(f) = X(f) \cdot H(f)$
- In particolare, $Y(f)$ viene: $\begin{cases} \text{AMPLIFICATA del FATTORE } |H(f)| \\ \text{SFASATA dell'ANGOLO } \measuredangle H(f) \end{cases}$

Per questo i SISTEMI LTI vengono chiamati FILTRI (LINEARI)

FILTRI IDEALI

SISTEMI SENZA DISTORSIONE (DISTORTIONLESS)

- Un SISTEMA LTI è detto SENZA DISTORSIONE sse:

l'USCITA è UGUALE all'INGRESSO, A MENO DI: $\begin{cases} \text{- UN FATTORE COSTANTE} \\ \text{- UN RITARDO} \end{cases}$

FORMALMENTE, dato $S[\cdot]$: $x(t) \rightarrow \boxed{\begin{matrix} S[\cdot] \text{ LTI} \\ h(t) \,/\, H(f) \end{matrix}} \rightarrow y(t)$

$S[\cdot]$ è SENZA DISTORSIONE sse: $y(t) = A\, x(t - t_0)$

→ RISPOSTA all'IMPULSO: $h(t) = A\,\delta(t - t_0)$

$\downarrow \mathcal{F}$ $\qquad \downarrow \mathcal{F}$

RISPOSTA in FREQUENZA: $H(f) = A\, e^{-j2\pi f t_0}$

Osserviamo $H(f)$: $\begin{cases} |H(f)| = A : \text{amplificazione COSTANTE}, \forall f \\ \measuredangle H(f) = -2\pi t_0 f : \text{SFASAMENTO proporzionale a } f \end{cases}$ → RITARDO di t_0

Es : $\begin{cases} s(t) = \cos(2\pi f t - \varphi) \\ \varphi = kf \end{cases} \longrightarrow \cos(2\pi f t - kf) = \cos\left(2\pi f\left(t - \frac{k}{2\pi}\right)\right)$

RITARDO

FILTRO IDEALE : $\begin{cases} \text{un SISTEMA SENZA DISTORSIONE per } f \in B \\ \text{un SISTEMA con } H(f) = 0 \text{ per } f \notin B \end{cases}$

$$H(f) = \begin{cases} A e^{-j2\pi f t_0} & \text{per } f \in B \\ 0 & \text{per } f \notin B \end{cases}$$

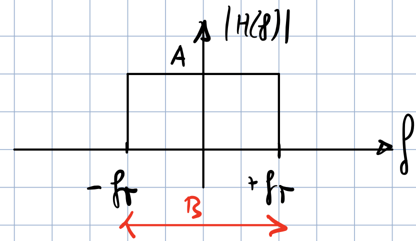
B : BANDA PASSANTE di un FILTRO IDEALE

In funzione di **come è DEFINITA B** ottengo diverse **TIPOLOGIE di FILTRO**

FILTRO PASSA-BASSO IDEALE (LOW-PASS) : $B : -f_T \le f \le f_T$

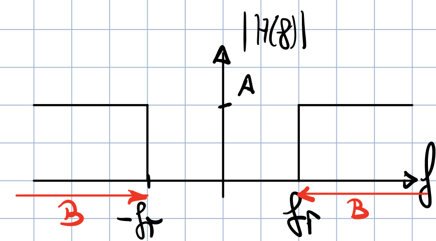
f_T : FREQUENZA di TAGLIO

$$H_{LP}(f) = \begin{cases} A e^{-j2\pi f t_0} & , |f| \le f_T \\ 0 & \text{altrove} \end{cases} = A e^{-j2\pi f t_0} \, rect\left(\frac{f}{2f_T}\right)$$



FILTRO PASSA-ALTO IDEALE (HIGH-PASS) : $B: |f| \ge f_T$

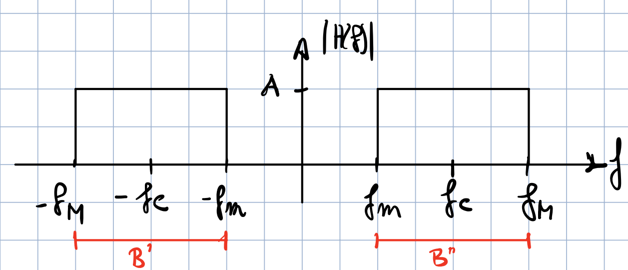
$$H_{HP}(f) = \begin{cases} A e^{-j2\pi f t_0} & |f| \ge f_T \\ 0 & \text{altrove} \end{cases} = A e^{-j2\pi f t_0}\left[1 - rect\left(\frac{f}{2f_T}\right)\right]$$



FILTRO PASSA-BANDA IDEALE (BAND-PASS): $B : f_m \le |f| \le f_M$

$$H_{BP}(f) = \begin{cases} A e^{-j2\pi f t_0} & f_m \le |f| \le f_M \\ 0 & \text{altrove} \end{cases} = \begin{cases} f_c = \frac{f_m + f_M}{2} \\ B = f_M - f_m \end{cases}$$

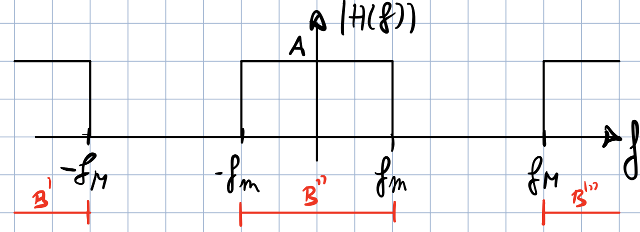
$$= \left[rect\left(\frac{f - f_c}{B}\right) + rect\left(\frac{f + f_c}{B}\right)\right] A e^{-j2\pi f t_0}$$



FILTRO ARRESTA-BANDA IDEALE (BAND-STOP) : $B: |f| \le f_m \vee |f| \ge f_M$

$$H_{BS}(f) = \begin{cases} A e^{-j2\pi f t_0} & |f| \le f_m \vee |f| \ge f_M \\ 0 & \text{altrove} \end{cases} =$$

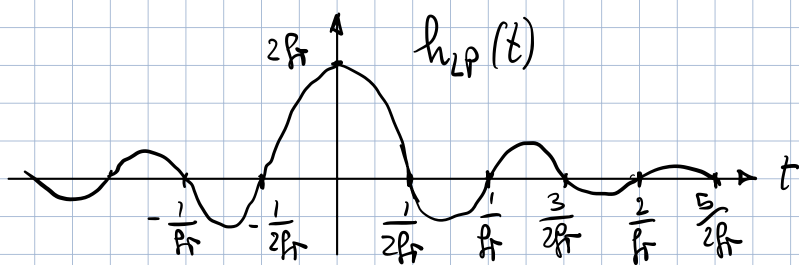
$$= 1 - H_{BP}(f)$$



Nel dominio dei TEMPI: **RISPOSTA all'IMPULSO dei FILTRI IDEALI**

PASSA-BASSO

$$H_{LP}(f) = \begin{cases} 1 & |f| \le f_T \\ 0 & \text{altrove} \end{cases} = \text{rect}\left(\frac{f}{2f_T}\right) \xrightarrow{\mathcal{F}^{-1}} h_{LP}(t) = 2f_T \,\text{sinc}(2f_T \cdot t)$$



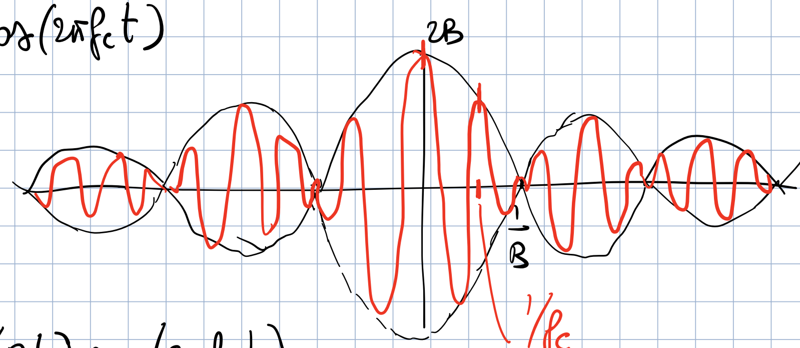
PASSA-ALTO

$$H_{HP}(f) = \begin{cases} 0 & |f| \le f_T \\ 1 & \text{altrove} \end{cases} = 1 - \text{rect}\left(\frac{f}{2f_T}\right) \xrightarrow{\mathcal{F}^{-1}} h_{HP}(t) = \delta(t) - 2f_T \,\text{sinc}(2f_T \cdot t)$$

PASSA-BANDA

$$H_{BP}(f) = \begin{cases} 1 & f_m \le |f| \le f_M \\ 0 & \text{altrove} \end{cases} = \text{rect}\left(\frac{f-f_c}{B}\right) + \text{rect}\left(\frac{f+f_c}{B}\right) = \text{rect}\left(\frac{f}{B}\right) * \left[\delta(f-f_c) + \delta(f+f_c)\right]$$

$$\xrightarrow{\mathcal{F}^{-1}} h_{BP}(t) = B\,\text{sinc}(Bt) \cdot \Big[\underbrace{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}_{2\cos(2\pi f_c t)}\Big] = 2B\,\text{sinc}(Bt)\cos(2\pi f_c t)$$

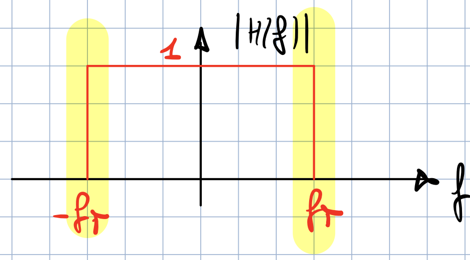


ARRESTA-BANDA

$$H_{SB}(f) = 1 - H_{BP}(f)$$

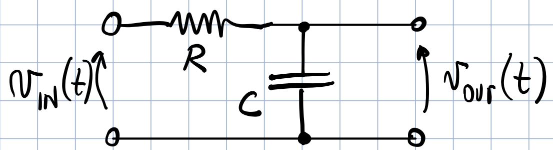
$$\xrightarrow{\mathcal{F}^{-1}} h_{SB}(t) = \delta(t) - 2B\,\text{sinc}(Bt)\cos(2\pi f_c t)$$

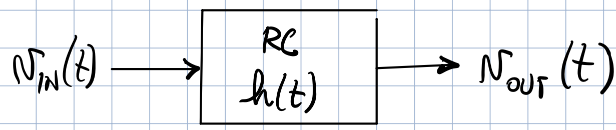
FILTRI REALI



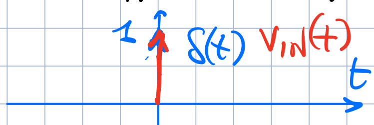
Esempio di PASSA-BASSO REALE :

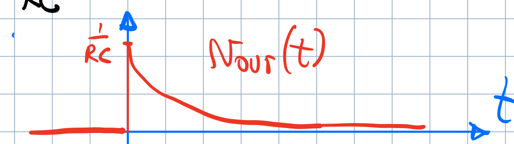
IL CIRCUITO RC





$$v_{IN}(t) = \delta(t) \qquad\qquad v_{OUT}(t) = \frac{1}{RC} u(t) e^{-\frac{t}{RC}} = h(t)$$





$$h(t) = \frac{1}{RC} u(t) e^{-\frac{t}{RC}}$$

FOURIER PAIR : $e^{-at} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a + j2\pi f}$

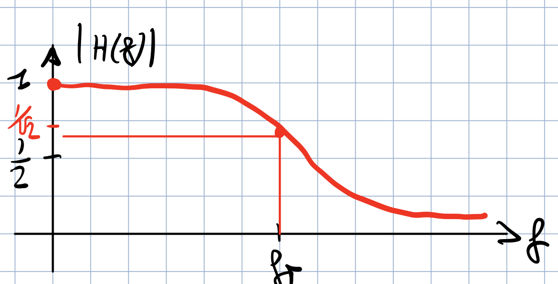
$$\left[a = \frac{1}{RC}\right] \rightarrow \quad H(f) = \frac{1}{RC} \frac{1}{\frac{1}{RC} + j2\pi f} = \frac{1}{1 + j2\pi RCf}$$

Definisco : $\frac{1}{2\pi RC} = f_T \longrightarrow \boxed{H(f) = \frac{1}{1 + j\,f/f_T}}$

Risposte in AMPIEZZA e FASE :

$$|H(f)| = \frac{|1|}{|1 + j\,f/f_T|} = \frac{1}{\sqrt{1^2 + \left(\frac{f}{f_T}\right)^2}} = \begin{cases} f \ll f_T \longrightarrow |H(f)| \approx 1 \\ f = f_T \longrightarrow |H(f)| = 1/\sqrt{2} \\ f \gg f_T \longrightarrow |H(f)| \approx f_T/f \end{cases}$$

$$\measuredangle H(f) = \cancel{\measuredangle(1)}_{0} - \measuredangle\left(1 + j\frac{f}{f_T}\right) = -\,\mathrm{atan2}\left(\frac{f}{f_T}\right) = \begin{cases} f \ll f_T \longrightarrow \measuredangle H(f) \approx 0 \\ f = f_T \longrightarrow \measuredangle H(f) = -\pi/4 \\ f \gg f_T \longrightarrow \measuredangle H(f) \approx -\pi/2 \end{cases}$$



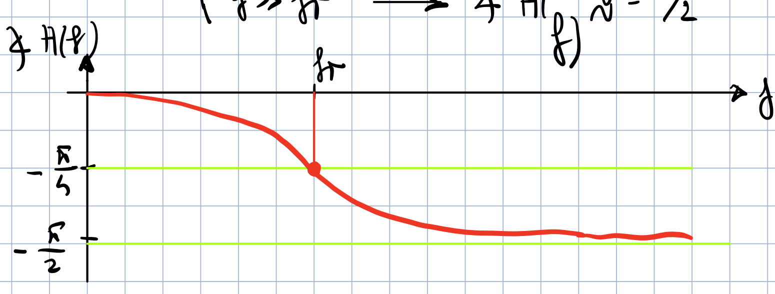
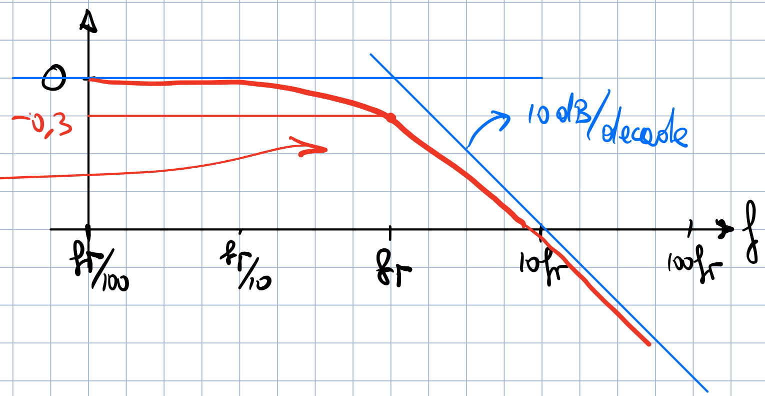


DIAGRAMMA di BODE : GRAFICO "LOG - LOG" di $|H(f)|$

$\log_{10} |H(f)|$ vs $\log_{10} f$

FILTRO del I ORDINE



FAMIGLIE di FILTRI REALI :

- filtri di BUTTERWORTH
- filtri di CHEBYSHEV
- filtri di BESSEL

ESERCIZI di RIEPILOGO

Calcolare la $H(f)$ di un filtro ideale PASSA-ALTO con $f_T = 100\,Hz$, $G = 8$ e ritardo $\tau = 0{,}5\,s$.

$$\rightarrow \quad H(f) = \begin{cases} 8\, e^{-j2\pi f 0{,}5} & |f| > 100 \\ 0 & \text{altrove} \end{cases} = 8\left[1 - rect\left(\frac{f}{200}\right)\right] e^{-j\pi f}$$

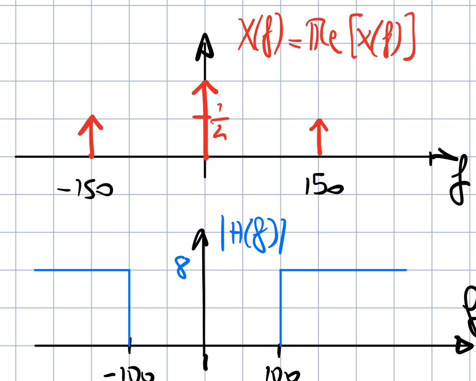
... e calcolare l'uscita del filtro per l'ingresso :

$$x(t) = \cos^2(150\pi t) = \cos(150\pi t)\cdot\cos(150\pi t)$$

$\mathcal{F} \downarrow$

$$X(f) = \frac{1}{2}\left[\delta(f+75) + \delta(f-75)\right] * \frac{1}{2}\left[\delta(f+75) + \delta(f-75)\right] =$$

$$= \frac{1}{4}\left[\delta(f+150) + 2\delta(f) + \delta(f-150)\right]$$



$$Y(f) = H(f)\,X(f) = \begin{cases} X(f)\; 8 e^{-j\pi f} & |f| > 100 \\ 0 & \text{altrove} \end{cases} =$$

$$= 8 e^{-j\pi f}\left[\frac{1}{4}\delta(f+150) + \frac{1}{4}\delta(f-150)\right] = 2\,\delta(f-150)\, e^{-j\pi f} + 2\,\delta(f+150)\, e^{-j\pi f} =$$

$$= 2\,\delta(f-150)\, \underbrace{e^{-j\pi 150}}_{1} + 2\,\delta(f+150)\, \underbrace{e^{+j\pi 150}}_{1} = 2\left[\delta(f-150) + \delta(f+150)\right]$$

$$\xrightarrow{\mathcal{F}^{-1}} \quad \boxed{y(t) = 4\cos(300\pi t)}$$