
TRASFORMATE per SEGNALI DIGITALI

TRASFORMATA di FOURIER per SEGNALI TEMPO-DISCRETI

Dato $s(t)$ TEMPO-CONTINUO → $s_c(t)$ CAMPIONATO (TEMPO-DISCRETO)

$$s_c(t) = s(t)\sum_{n=-\infty}^{\infty} \delta(t-nT_s) \xleftrightarrow{\mathcal{F}} S_c(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} S\left(f-\frac{n}{T_s}\right)$$

PERIODICO, "PERIODO": $\frac{1}{T_s} = f_s$

Riusciamo ad esprimere $S_c(f)$ in FUNZIONE solo dei CAMPIONI $s(nT_s)$?

$$s_c(t) = s(t)\sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \sum_{n=-\infty}^{\infty} s(nT_s)\,\delta(t-nT_s)$$

Calcolo lo SPETTRO di $s_c(t)$ partendo da:

$$s_c(t) \xrightarrow{\mathcal{F}} S_c(f) = \int_{-\infty}^{\infty} s_c(t)\,e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} s(nT_s)\,\delta(t-nT_s)\,e^{-j2\pi ft}\,dt =$$

$$= (\text{x linearità}) = \sum_{n=-\infty}^{\infty} s(nT_s)\underbrace{\int_{-\infty}^{\infty}\delta(t-nT_s)\,e^{-j2\pi ft}\,dt}_{e^{-j2\pi fnT_s}} = \sum_{n=-\infty}^{\infty} s(nT_s)\,e^{-j2\pi fnT_s}$$

$$s(nT_s) \equiv s_c(t) \xleftrightarrow{\mathcal{F}} S_c(f) = \sum_{n=-\infty}^{\infty} s(nT_s)\,e^{-j2\pi fnT_s}$$

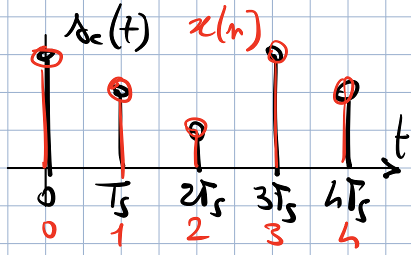
Verifico che $S_c(f)$ sia PERIODICA, con PERIODO $\frac{1}{T_s}=f_s$: $f_sT_s=1$

$$S_c(f+kf_s) = \sum_{-\infty}^{\infty}{}_n\, s(nT_s)\,e^{-j2\pi(f+kf_s)nT_s} = \underbrace{\sum_{-\infty}^{\infty}{}_n\, s(nT_s)\,e^{-j2\pi f nT_s}}_{S_c(f)}\; e^{-j2\pi k f_s n T_s} \to 1$$

$$= S_c(f) \quad \checkmark$$

Cerchiamo una versione "NORMALIZZATA" di $S_c(f)$ (rispetto a f_s)

Considero : $x(n) = s_c(nT_s)$



$$x(n) \longrightarrow S_c(f) = \sum_{-\infty}^{\infty}{}_n\, x(n)\,e^{-j2\pi f nT_s}$$

DEFINISCO la FREQUENZA NORMALIZZATA (rispetto a f_s):

$$\varphi = f/f_s = f\cdot T_s \quad \text{Freq. NORMALIZZATA}$$

$$\to S_c(f) = \sum_{-\infty}^{\infty}{}_n\, x(n)\,e^{-j2\pi n\varphi} = X(\varphi)\,; \quad 0\le\varphi<1$$

PERIODICA con PERIODO = 1

$$\boxed{x(n),\ n\in\mathbb{Z} \quad \xleftrightarrow{\mathcal{F}} \quad X(\varphi) = \sum_{-\infty}^{\infty}{}_n\, x(n)\,e^{-j2\pi n\varphi}, \quad 0\le\varphi<1}$$

$X(\varphi)$: TRASFORMATA di FOURIER a TEMPO DISCRETO /
DISCRETE-TIME FOURIER TRANSFORM : DTFT

$X(\varphi)$ è PERIODICA (periodo φ: 1) → SERIE di FOURIER

$$X(\varphi) = \sum_k c_k\, e^{j2\pi \frac{n}{1}\varphi} \longrightarrow \begin{cases} x(n) = c_k \\ \text{FASORI INVERTITI} \end{cases}$$

Formula di ANALISI in SF → formula di SINTESI ≡ DTFT^{-1}

$$\{c_k\} \equiv x(n) = \frac{1}{T}\int_T X(\varphi)\,e^{j2\pi n\varphi}\,d\varphi = (T=1) = \int_0^1 X(\varphi)\,e^{j2\pi n\varphi}\,d\varphi$$

TRASFORMATA di FOURIER a TEMPO DISCRETO (DTFT)

Formula di ANALISI : $x(n),\ n\in\mathbb{Z} \xrightarrow{\text{DTFT}} X(\varphi) = \sum_{-\infty}^{\infty}{}_n\, x(n)\,e^{-j2\pi n\varphi}$

Formula di SINTESI: $X(\varphi),\ \varphi\in\mathbb{R},\ 0\le\varphi<1 \xrightarrow{\text{DTFT}^{-1}} x(n) = \int_0^1 X(\varphi)\,e^{j2\pi n\varphi}\,d\varphi$

PROPRIETÀ delle DTFT :

TRASLAZIONE nei TEMPI (DISCRETI)

$$x(n) \leftrightarrow X(\varphi) \longrightarrow x(n-m) \longleftrightarrow X(\varphi)\, e^{-j2\pi\varphi m}$$

DIM: $x(n-m) \xrightarrow{DTFT} X'(\varphi) = \sum_{-\infty}^{\infty}{}_{n}\, x(n-m)\, e^{-j2\pi\varphi n} = (n' = n-m) =$

$$= \sum_{-\infty}^{\infty}{}_{n'}\, x(n')\, e^{-j2\pi(n'+m)\varphi} = \underbrace{\sum_{-\infty}^{\infty}{}_{n'}\, x(n)\, e^{-j2\pi n'\varphi}}_{X(\varphi)} \cdot \underbrace{e^{-j2\pi m\varphi}}_{COSTANTE} = X(\varphi)\, e^{-j2\pi\varphi m} \quad \checkmark$$

TRASLAZIONE nelle FREQUENZE

$$x(n) \overset{DTFT}{\longleftrightarrow} X(\varphi) \longrightarrow x(n)\, e^{j2\pi n\varphi_0} \overset{DTFT}{\longleftrightarrow} X(\varphi-\varphi_0)$$

CONVOLUZIONE

$$\begin{cases} x(n) \overset{DTFT}{\longleftrightarrow} X(\varphi) \\ y(n) \longleftrightarrow Y(\varphi) \end{cases} \Longrightarrow x(n) * y(n) \overset{DTFT}{\longleftrightarrow} X(\varphi)\cdot Y(\varphi)$$

$s(n)$ DISCRETO $\xrightarrow{DTFT}$ $S(f)$ PERIODICO

$s(t)$ PERIODICO $\xrightarrow{SF}$ $S(f) = \{c_k\}$ DISCRETO

$x(n)$ DISCRETO e PERIODICO $\longrightarrow$ $X(\varphi_k)$ è PERIODICA e DISCRETA

↳ $\varphi_k ,\ k \in \mathbb{Z}$

TRASFORMATA di FOURIER di un SEGNALE DIGITALE

Segnale digitale : SEQUENZA di N CAMPIONI

Sia dato $s(n)$ di N CAMPIONI : $n = 0, \ldots, N-1$

↳ ne definiamo una versione PERIODICIZZATA $x(n)$

$$s(n),\ n = 0, 1, \ldots, N-1 \longrightarrow x(n):\ x(n+kN) = s(n),\ \forall k \in \mathbb{Z}$$

↳ è DISCRETO e PERIODICO !

$x(n)$ è DISCRETO $\xrightarrow{DTFT}$ $X(\varphi) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi n\varphi}$, $0 \leq \varphi < 1$

$x(n)$ è PERIODICO $\xrightarrow{FS}$ $x(n) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{N} n}$ $\{c_k\}$ è PERIODICA, PERIODO $= N$

Considero: $x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi \frac{k}{N} n}$, $n = 0, 1, \dots, N-1$

Posso esprimere i $\{c_k\}$ con la formula di ANALISI della FS:

$c_k = \frac{1}{T} \int_T x(t) e^{-j2\pi \frac{k}{N} t} dt$, dove nel ns. caso: $\begin{cases} \bullet\ x(t) \rightarrow x(n) \\ \bullet\ \text{PERIODO } T = N \end{cases}$

Quindi: $c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi \boxed{\frac{k}{N}} n}$, $0 \leq \varphi = \frac{k}{N} < 1 \rightarrow k = 0, 1, \dots, N-1$

$0 \leq \varphi < 1$ $= \frac{1}{N} X'\left(\varphi = \frac{k}{N}\right)$

Definisco: $\sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{k}{N} n} = N c_k = X'\left(\varphi = \frac{k}{N}\right) = X_{DFT}(k) = \text{DFT}\{x(n)\}$

DISCRETE FOURIER TRANSFORM - DFT

Dato un SEGNALE DISCRETO di N CAMPIONI : $x(n)$, $n = 0, 1, \dots, N-1$
(considerato PERIODICIZZATO con periodo $= N$)

$x(n) \xrightarrow{DFT} X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{k}{N} n}$, $n, k = 0, 1, \dots, N-1$

e la DFT INVERSA (IDFT, DFT^{-1}):

$X(k) \xrightarrow{IDFT} x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi \frac{k}{N} n}$, $n, k = 0, \dots, N-1$

★ DFT ci dà una RELAZIONE BIUNIVOCA tra SEQUENZE di N CAMPIONI

★ la DFT CORRISPONDE alla DTFT del segnale PERIODICIZZATO

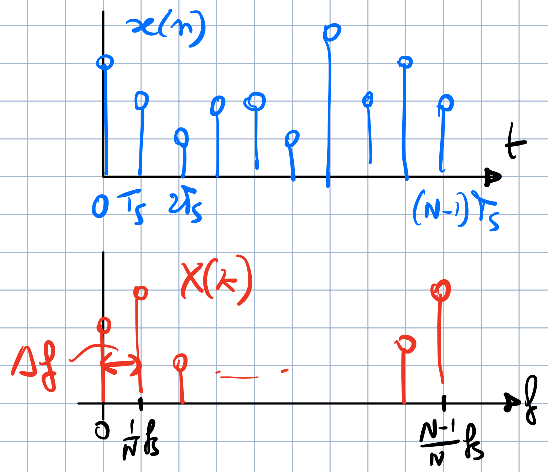
$x(n) \overset{DFT}{\longleftrightarrow} X(k)$ $n, k = 0, \dots, N-1$

$x(n)$ PERIODICIZZATO $\overset{DTFT}{\longleftrightarrow}$ $X(k)$ PERIODICIZZATA : $n, k \in \mathbb{Z}$

RISOLUZIONE e DURATA nella DFT :

Considero un segnale di N CAMPIONI, CAMPIONATO ogni T_s $\left(f_s = \frac{1}{T_s}\right)$

	RISOLUZIONE	DURATA
nei TEMPI	T_s	$N T_s$
nelle FREQUENZE	$\Delta f = \frac{f_s}{N} = \frac{1}{N T_s}$	f_s



ESEMPIO: segnale campionato a $f_s = 48$ kHz

→ VOGLIO una DFT con RISOLUZIONE $\Delta f = 10$ Hz

↳ COME FARE la DFT ?

$$\Delta f = 10 \text{ Hz} = \frac{1}{N T_s} = \frac{f_s}{\boxed{N}} \longrightarrow N = \frac{f_s}{10 \text{ Hz}} = 4800 \text{ campioni}$$

$$\rightarrow \text{DURATA} = N T_s = \frac{1}{10} \text{ s} = 0{,}1 \text{ s}$$

CALCOLO della DFT in FORMA MATRICIALE

Date le espressioni della DFT:

$$\begin{cases} X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{kn}{N}} \\ x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi\frac{kn}{N}} \end{cases} \qquad n,k = 0,..,N-1$$

Posso esprimere DFT e IDFT come PRODOTTO MATRICE-VETTORE:

DFT:

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = k\!\updownarrow \underbrace{\overset{\longleftarrow\; n \;\longrightarrow}{\begin{bmatrix} W_{nk} = e^{-j2\pi\frac{nk}{N}} \\ n,k = 0,..,N-1 \end{bmatrix}}}_{F} \cdot \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \longrightarrow \underline{X} = F \cdot \underline{x}$$

IDFT:

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \frac{1}{N} \overbrace{\begin{bmatrix} W_{nk} = e^{j2\pi\frac{nk}{N}} \\ n,k = 0,..,N-1 \end{bmatrix}}^{F^{-1}} \cdot \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \longrightarrow \underline{x} = \frac{1}{N} F^{-1} \cdot \underline{X}$$

PROPRIETÀ : $F^{-1} = \frac{1}{N}\bar{F}$

COMPLESSITÀ di CALCOLO della DFT : $O(N^2)$

Esiste un ALGORITMO OTTIMIZZATO :

FAST FOURIER TRANSFORM - FFT : COMPLESSITÀ : $O(N \log_2 N)$

ESEMPIO : calcolo matriciale della DFT con $N=4$

↳ calcolo $F_{4\times4}$

$$F_{4\times4} = \begin{bmatrix} W_{nk} = e^{-j2\pi\frac{nk}{4}} \\ n,k = 0,\dots,3 \end{bmatrix} = \begin{bmatrix} W_{nk} = e^{-j\frac{\pi}{2}nk} \\ n,k = 0..3 \end{bmatrix} = \begin{array}{c|cccc} & 0 & \overset{n}{1} & 2 & 3 \\ \hline 0 & 1 & 1 & 1 & 1 \\ k\ 1 & 1 & e^{-j\frac{\pi}{2}} & e^{-j\pi} & e^{-j\frac{3}{2}\pi} \\ 2 & 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} \\ 3 & 1 & e^{-j\frac{3}{2}\pi} & e^{-j3\pi} & e^{-j\frac{9}{2}\pi} \end{array} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$$

Calcoliamo la DFT del segnale $x(n) = \{\underset{\uparrow}{1}, 1, 0, 0\}$

$$\underline{X(k)} = F \cdot \underline{x(n)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+1+0+0=2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix} = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix} = X(k)$$

IDFT : calcolo $F^{-1} = \frac{1}{N}\bar{F} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$

$$x(n) = F^{-1} \cdot X(k) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}\begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2+1\cancel{-j}+1\cancel{+j} = 4 \\ 2\cancel{+j}+1\cancel{-j}+1 = 4 \\ 2-1\cancel{+j}-1\cancel{-j} = 0 \\ 2\cancel{-j}-1\cancel{+j}-1 = 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

PROPRIETÀ della DFT

TRASLAZIONE CICLICA nei TEMPI :

$$x(n) \xleftrightarrow{DFT} X(k) \longrightarrow x(\langle n-a \rangle_N) \xleftrightarrow{DFT} X(k)e^{-j2\pi\frac{ak}{N}}$$

TRASLAZIONE CICLICA nelle FREQUENZE :

$$x(n) \xleftrightarrow{DFT} X(k) \longrightarrow x(n)e^{j2\pi\frac{h}{N}n} \xleftrightarrow{DFT} X(\langle k-h \rangle_N)$$

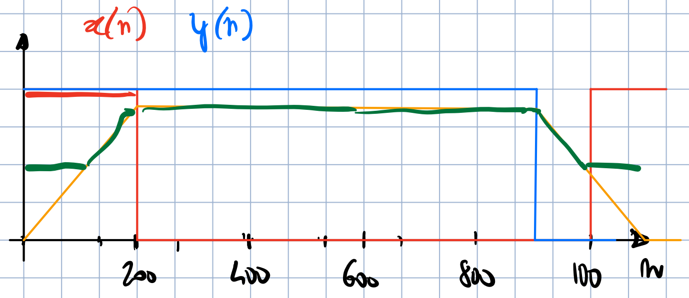
CONVOLUZIONE DISCRETA CICLICA

Def: $x(n) \circledast y(n) = \sum_{0}^{N-1}{}_i\ x(i)\, y(\langle n-i \rangle_N) \qquad n = 0, \dots, N-1$

$$\begin{array}{l} x(n) \xleftrightarrow{DFT} X(k) \\ y(n) \xleftrightarrow{DFT} Y(k)\end{array} \longrightarrow \quad x(n)\circledast y(n) \xleftrightarrow{DFT} X(k)\cdot Y(k)$$

Esempio: $\quad x(n) = rect\left(\frac{n-100}{200}\right)$

N=1000 $\quad y(n) = rect\left(\frac{n-450}{900}\right)$



TRASFORMATA Z

Dato un segnale $x(n)$, $n \in \mathbb{Z}$:

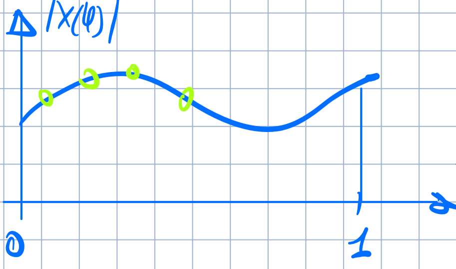
$$x(n) \xrightarrow{DTFT} X(\varphi) = \sum_{-\infty}^{+\infty}{}_n\ x(n)\, e^{-j2\pi n\varphi}, \quad 0 \le \varphi < 1$$

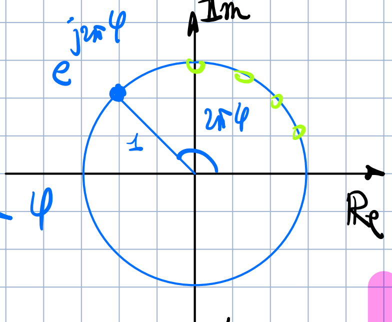
Considero che: $e^{-j2\pi n\varphi} = f(n) = (\underbrace{e^{j2\pi\varphi}}_{z\in\mathbb{C}})^{-n} = z^{-n}$

Ponendo: $z = e^{j2\pi\varphi}$

$$x(n) \xrightarrow{DTFT} \sum_{-\infty}^{\infty}{}_n\ x(n)\, e^{-j2\pi n\varphi} = \sum_{-\infty}^{\infty}{}_n\ x(n)\, z^{-n} = X(z)$$

TRASFORMATA Z di $x(n)$





TRASFORMATA Z:

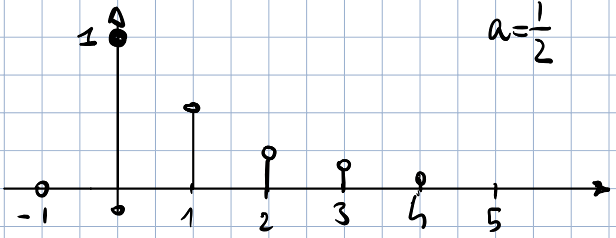
Dato un SEGNALE DISCRETO $x(n)$, $n\in\mathbb{Z}$ si definisce TRASFORMATA Z di $x(n)$ la funzione della VARIABILE COMPLESSA z:

$$x(n) \xleftrightarrow{Z} X(z) = \sum_{-\infty}^{\infty}{}_n\ x(n)\, z^{-n}, \quad n\in\mathbb{Z},\ z\in\mathbb{C}$$

Il LUOGO dei PUNTI $z\in\mathbb{C}$ per i quali $X(z)$ CONVERGE è detto la REGIONE di CONVERGENZA (ROC) di $X(z)$. Affinché la TRASFORMATA $X(z)$ sia UNIVOCAMENTE DETERMINATA, è necessario definire:

- ★ la FUNZIONE $X(z)$;
- ★ la sue REGIONE di CONVERGENZA (ROC)

Esempio di calcolo di $\mathcal{Z}\{\ \}$:



$$x(n) = a^n u(n) = \begin{cases} a^n, & n \geq 0 \\ 0 & n < 0 \end{cases}$$

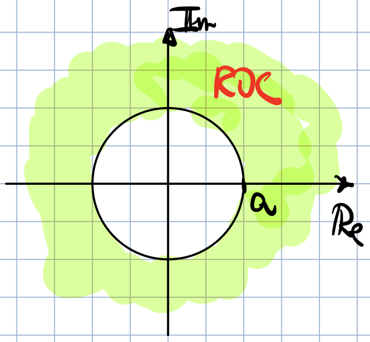
$$X(z) = \sum_{-\infty}^{\infty}{}_n\, x(n) z^{-n} = \sum_{0}^{\infty}{}_n\, a^n z^{-n} = \sum_{0}^{\infty}{}_n \left(\frac{a}{z}\right)^n$$

[SERIE di POTENZE: $\sum_{0}^{\infty}{}_n\, \alpha^n \longrightarrow$ CONVERGE per $|\alpha| < 1$: $\sum_{0}^{\infty}{}_n\, \alpha^n = \frac{1}{1-\alpha}$]

Quindi:

→ se $\left|\frac{a}{z}\right| < 1 \rightarrow X(z) = \sum_{0}^{\infty}{}_n \left(\frac{a}{z}\right)^n$ CONVERGE: $X(z) = \frac{1}{1-\frac{a}{z}} = \frac{1}{1-az^{-1}}$

→ se $\left|\frac{a}{z}\right| \geq 1 \longrightarrow X(z) = \sum_{0}^{\infty}{}_n \left(\frac{a}{z}\right)^n$ DIVERGE: $\nexists\, X(z)$



$\boxed{\text{ROC}}$: $\left|\frac{a}{z}\right| < 1 \longrightarrow \boxed{|z| > |a|}$

$$x(n) = a^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \frac{1}{1-az^{-1}} \ ; \ \text{ROC}: |z| > |a|$$

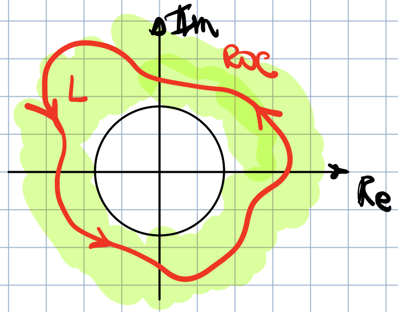
ANTITRASFORMATA Z:

Data $X(z) = \mathcal{Z}\{x(n)\}$, si ottiene $x(n)$ a partire da $X(z)$ con:

$$X(z) \xrightarrow{\mathcal{Z}^{-1}} x(n) = \frac{1}{j2\pi} \oint_L X(z) z^{n-1} dz, \quad n \in \mathbb{Z}, \ z \in \mathbb{C}, \ L \in \text{ROC}[X(z)]$$

dove L è un QUALUNQUE PERCORSO CHIUSO che

- CONTIENE l'ORIGINE;
- è INTERAMENTE CONTENUTO in ROC[X(z)];
- è PERCORSO in senso ANTIORARIO



PROPRIETÀ delle TRASFORMATA Z

LINEARITÀ

$$\begin{matrix} x(n) \xleftrightarrow{Z} X(z) \\ y(n) \xleftrightarrow{Z} Y(z) \end{matrix} \longrightarrow a x(n) + b y(n) \xleftrightarrow{Z} a X(z) + b Y(z)$$

ROC: $ROC[X] \wedge ROC[Y]$

TRASLAZIONE nei TEMPI

$$x(n) \longleftrightarrow X(z) \; ; \; ROC[X] \longrightarrow x(n-a) \xleftrightarrow{Z} X(z)\, z^{-a}$$

ROC: $ROC[X], \begin{cases} z \neq 0, & a > 0 \\ z \neq \pm\infty & a < 0 \end{cases}$

CONVOLUZIONE

CONVOLUZIONE LINEARE

$$\begin{matrix} x(n) \xleftrightarrow{Z} X(z) \\ y(n) \xleftrightarrow{Z} Y(z) \end{matrix} \longrightarrow x(n) * y(n) \longleftrightarrow X(z) \cdot Y(z)$$

ROC: $ROC[X] \wedge ROC[Y]$

DERIVAZIONE

$$x(n) \xleftrightarrow{Z} X(z) \longrightarrow n \cdot x(n) \xleftrightarrow{Z} -z \frac{dX(z)}{dz}$$

ROC: $ROC[X]$

COPPIE NOTEVOLI :

IMPULSO DISCRETO

$$x(n) = \delta(n) \xleftrightarrow{Z} X(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = 1$$

ROC : $z \in \mathbb{C}$

IMPULSO TRASLATO

$$x(n) = \delta(n-m) \xleftrightarrow{Z} X(z) = \sum_{n=-\infty}^{\infty} \delta(n-m) z^{-n} = z^{-m},$$

ROC: $z \in \mathbb{C}, \begin{cases} z \neq 0 & m > 0 \\ z \neq \pm\infty & m < 0 \end{cases}$

SEQUENZA ESPONENZIALE

$$x(n) = a^{n} u(n) \xleftrightarrow{Z} X(z) = \sum_{n=0}^{\infty} a^{n} z^{-n} = \frac{1}{1 - a z^{-1}},$$

ROC : $|z| > |a|$

GRADINO UNITARIO

$$x(n) = u(n) \xleftrightarrow{Z} X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} \; ;$$

ROC : $|z| > 1$

CALCOLO delle TRASF. Z : CASI COMUNI :

★ Sequenze ESPONENZIALI : $x(n) = a^n u(n) \xleftrightarrow{Z} X(z) = \frac{1}{1-az^{-1}}$, ROC : $|z| > |a|$

Per LINEARITÀ :

$$x(n) = \sum_i c_i a_i^n u(n) \xleftrightarrow{Z} X(z) = \sum_i \frac{c_i}{1-a_i z^{-1}}$$

ROC : $\bigwedge \{ |z| > |a_i| \} \equiv$
$\equiv |z| > |a_{i_{MAX}}|$

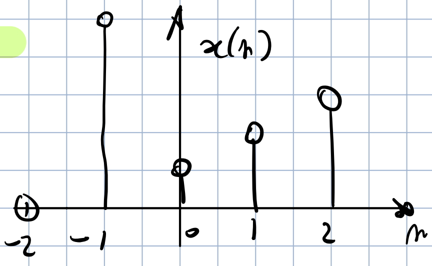
★ Sequenze a LUNGHEZZA FINITA :

$$x(n) = \begin{cases} a_n , & n_0 \leq n \leq n_1 \\ 0 & \text{altrove} \end{cases} ; \quad x(n) = \{ a_{n_0}, a_{n_0+1}, \dots, a_{n_1} \}$$

Possiamo scrivere : $x(n) = \{ a_{n_0} \dots a_{n_1} \} = \sum_{n_0}^{n_1} {}_i \, a_i \, \delta(n-i)$

$$x(n) = \sum_{n_0}^{n_1} {}_i \, a_i \, \delta(n-i) \xleftrightarrow{Z} \sum_{n_0}^{n_1} {}_i \, a_i \, Z\{\delta(n-i)\} = \sum_{n_0}^{n_1} {}_i \, a_i z^{-i}$$

$$x(n) = \{ a_{n_0}, \dots, a_{n_1} \} \xleftrightarrow{Z} X(z) = \sum_{n_0}^{n_1} {}_i \, a_i z^{-i}$$

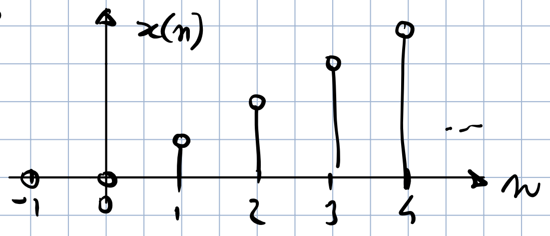


Esempio : calcolare $Z\{ x(n) = \{ \underset{n_0}{5}, \underset{\uparrow}{1}, 2, \underset{n_1}{3} \} \}$

$$X(z) = \sum_{-1}^{2} {}_i \, a_i z^{-i} = 5z + 1 + 2z^{-1} + 3z^{-2}$$

→ ROC : $z \neq 0 ; z \neq \pm\infty$

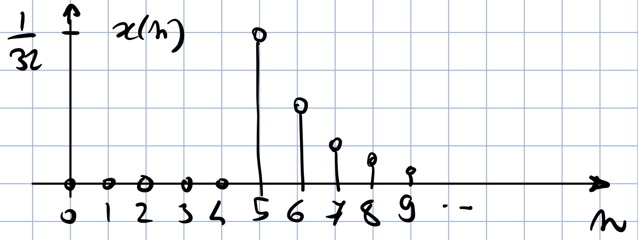
ESEMPI di calcolo :



★ $x(n) = n \cdot u(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$

$$X(z) = -z \frac{d}{dz} Z\{u(n)\} = -z \frac{d}{dz} \frac{1}{1-z^{-1}} = -z \frac{d}{dz} \frac{z}{z-1} = -z \frac{1(\cancel{z}-1) - \cancel{z}(1)}{(z-1)^2} =$$

$$= -z \frac{-1}{(z-1)^2} = \frac{z}{z^2-2z+1} = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}$$; ROC : ROC $\left[\frac{1}{1-z^{-1}} \right]$: $|z| > 1$



★ $x(n) = \begin{cases} \left(\frac{1}{2}\right)^n & \text{per } n \geq 5 \\ 0 & \text{per } n < 5 \end{cases}$

$$x(n) = \{ \underset{\uparrow}{0}, 0, 0, 0, 0, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \dots \}$$

TRASLAZIONE
$n' = n+5$

$$x(n) = \begin{cases} \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^{n-5} & \text{per } n \geq 5 \\ 0 & \text{altrove} \end{cases} \rightarrow n' = n-5 \rightarrow \cancel{x(n')} \overset{?}{=} \begin{cases} \frac{1}{32}\left(\frac{1}{2}\right)^{\cancel{n'}} , & n' \geq 0 \\ 0 , & \text{altrove} \end{cases} = \frac{1}{32}\left(\frac{1}{2}\right)^n u(n)$$

$x(n+5) =$

$n+5-5$

$$x(n') \overset{\mathcal{Z}}{\longleftrightarrow} X'(z) = \frac{1}{32}\,\frac{1}{1-\frac{1}{2}z^{-1}} = \frac{1}{32-16z^{-1}} \leftarrow$$

$$x(n) = x(n'+5) \longrightarrow X(z) = X'(z)\,z^{5} = \frac{z^{5}}{32-16z^{-1}} \qquad \text{ROC}: |z|>\frac{1}{2},\ z\neq\pm\infty$$

CALCOLO di ANTITRASFORMATA Z : CASI COMUNI

★ Sequenze di lunghezza FINITA :

$$X(z) = \sum_{n_0}^{n_1}{}_i\ a_i z^{-i} \overset{\mathcal{Z}}{\longleftrightarrow} x(n) = \{a_{n_0}, a_{n_0+1}, \dots, a_{n_1}\}$$

★ TRASFORMATE Z in FORMA RAZIONALE (RAPPORTO di POLINOMI)

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0+b_1z^{-1}+b_2z^{-2}+\dots+b_nz^{-M}}{1+a_1z^{-1}+a_2z^{-2}+\dots+a_Nz^{-N}} = \frac{B(z^{-1})}{A(z^{-1})}$$

← GRADO: M

← GRADO: N

Abbiamo 2 CASI POSSIBILI :

1) M < N : FRAZIONE PROPRIA

$$X(z) = \frac{B(z)}{1+a_1z^{-1}+\dots+a_Nz^{-N}} = \frac{B(z)}{\prod_{1}^{N}{}_i\,(1-p_iz^{-1})}$$

← N RADICI di A(z)

$$x^2-2x+1 = (x-1)(x-1)$$

Posso fare la SCOMPOSIZIONE in FRAZIONI PARZIALI

$$X(z) = \frac{B(z)}{\prod_{1}^{N}{}_i\,(1-p_iz^{-1})} = \frac{A_1}{1-p_1z^{-1}} + \frac{A_2}{1-p_2z^{-1}} + \dots + \frac{A_N}{1-p_Nz^{-1}} = \sum_{1}^{N}{}_i\ \frac{A_i}{1-p_iz^{-1}}$$

Se $X(z) = \sum_{1}^{N}{}_i \frac{A_i}{1-p_iz^{-1}} \xrightarrow{\mathcal{Z}^{-1}} x(n) = \sum_{1}^{N}{}_i\ A_i\, p_i^{n}\, u(n)$

2) FRAZIONE IMPROPRIA : M ≥ N

↳ scomporre in SOMMA di un POLINOMIO INTERO + FRAZIONE PROPRIA

$$X(z) = \frac{B_M(z^{-1})}{A_N(z^{-1})} = \underbrace{c_0+c_1z^{-1}+\dots+c_rz^{-r}}_{\text{SEQ. di LUNGHEZZA FINITA: } c_0\delta(n)+c_1\delta(n-1)+\dots+c_r\delta(n-r)} + \underbrace{\frac{B_{N-1}(z^{-1})}{A_N(z^{-1})}}_{\text{fraz. PROPRIA}}, \qquad r = M-N : \text{GRADO RELATIVO}$$

$$\xrightarrow{Z^{-1}} x(n) = c_0\delta(n) + c_1\delta(n-1) + \dots + c_r\delta(n-r) + \sum_{1}^{N}{}_i A_i\, p_i^n \cdot u(n)$$

ESEMPI:

$$X(z) = \frac{z^2}{1-3z+2z^2} = \frac{1}{2-3z^{-1}+z^{-2}} \begin{matrix} \rightarrow M=0 \\ \rightarrow N=2 \end{matrix} \rightarrow M<N : \text{Frazione PROPRIA}$$

FATTORIZZO il DENOMINATORE:

$$2-3z^{-1}+z^{-2} = \underbrace{2-2z^{-1}}-z^{-1}+z^{-2} = 2(1-z^{-1})-z^{-1}(1-z^{-1}) = (1-z^{-1})(2-z^{-1})$$

$$X(z) = \frac{1}{(1-z^{-1})(2-z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{2-z^{-1}} = \frac{A(2-z^{-1})+B(1-z^{-1})}{(1-z^{-1})(2-z^{-1})} \rightarrow \text{eguaglio i NUMERATORI:}$$

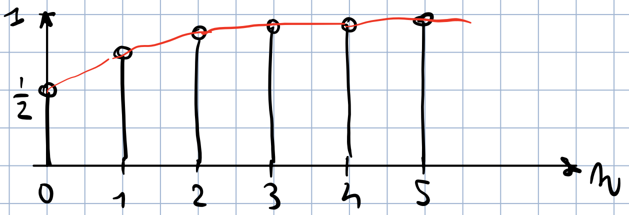
$$1 = A(2-z^{-1}) + B(1-z^{-1}) = 2A - Az^{-1} + B - Bz^{-1} = \underbrace{(2A+B)}_{1} - \underbrace{(A+B)}_{0} z^{-1}$$

$$\begin{cases} 2A+B=1 & 2A-A=1 \longrightarrow \\ A+B=0 \longrightarrow & B=-A \end{cases} \boxed{\begin{matrix} A=1 \\ B=-1 \end{matrix}}$$

Quindi: $$X(z) = \frac{1}{1-z^{-1}} - \frac{1}{2-z^{-1}} = \frac{1}{1-z^{-1}} - \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$\xrightarrow{z^{-1}} x(n) = u(n) - \frac{1}{2}\left(\frac{1}{2}\right)^n u(n) = \left[1-\left(\frac{1}{2}\right)^{n+1}\right]u(n) = \begin{cases} 1-\left(\frac{1}{2}\right)^{n+1} & n \geq 0 \\ 0 & n<0 \end{cases}$$

$$x(n) = \left\{ \underset{\uparrow}{\frac{1}{2}}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \dots \right\}$$



Calcolare l'antitrasformata di:

$$X(z) = \frac{1+2z^{-1}-z^{-2}+z^{-3}}{1-z^{-2}} \begin{matrix} \longrightarrow M=3 \\ \longrightarrow N=2 \end{matrix}$$

frazione IMPROPRIA → POL. INTERO + FRAZ. PROPRIA

$$= \frac{1-z^{-2}+2z^{-1}+z^{-3}}{1-z^{-2}} = \frac{1-z^{-2}}{1-z^{-2}} + \frac{2z^{-1}+z^{-3}}{1-z^{-2}} = 1 + \frac{2z^{-1}+z^{-3}}{1-z^{-2}} \begin{matrix} \rightarrow M=2 \\ \rightarrow N=2 \end{matrix}$$

$$= 1 + \frac{3z^{-1}\overbrace{-z^{-1}+z^{-3}}^{-z^{-1}(1-z^{-2})}}{1-z^{-2}} = \underbrace{1-z^{-1}}_{\text{POL. INTERO}} + \frac{3z^{-1}}{1-z^{-2}} \rightarrow \text{fraz. PROPRIA}$$

Scompongo in fraz. PARZIALI le fraz. proprie :

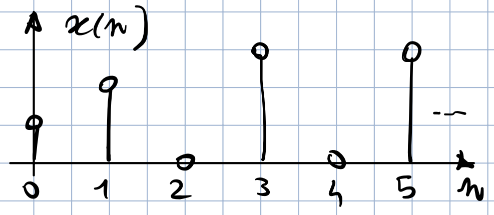
$$\frac{3z^{-1}}{1-z^{-2}} = \frac{3z^{-1}}{(1-z^{-1})(1+z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1+z^{-1}} = \frac{A(1+z^{-1})+B(1-z^{-1})}{(1-z^{-1})(1+z^{-1})}$$

$$3z^{-1} = A + Az^{-1} + B - Bz^{-1} = \underbrace{(A+B)}_{0} + \underbrace{(A-B)}_{3}z^{-1} \Rightarrow \begin{cases} A+B=0 \rightarrow A=-B \\ A-B=3 \rightarrow A+A=2A=3 \end{cases} \quad \boxed{\begin{array}{l} A=3/2 \\ B=-3/2 \end{array}}$$

$$\rightarrow \frac{3z^{-1}}{1-z^{-2}} = \frac{3}{2}\left[\frac{1}{1-z^{-1}} - \frac{1}{1+z^{-1}}\right] \xrightarrow{Z^{-1}} \frac{3}{2}\left[u(n) - (-1)^n u(n)\right]$$

$$\longrightarrow x(n) = \delta(n) - \delta(n-1) + \frac{3}{2}\left[1-(-1)^n\right]u(n)$$

$$x(n) = \{\underset{\uparrow}{1}, 2, 0, 3, 0, 3, 0, 3 \dots\}$$



SISTEMI DISCRETI LTI nel DOMINIO Z

$$\begin{tikzcd} x(n) \arrow[r] \arrow[d, "Z"'] & \boxed{\begin{array}{c}\text{SISTEMA LTI}\\ \text{DISCRETO:}\\ h(n)\end{array}} \arrow[r] \arrow[d, "Z"] & y(n) = x(n) * h(n) = \sum_{-\infty}^{\infty}{}_{i}\, x(i)h(n-i) \arrow[d] \\ X(z) & H(z) & \boxed{Y(z) = X(z)\cdot H(z)} \end{tikzcd}$$

$H(z) = Z\{h(n)\}$: FUNZIONE di TRASFERIMENTO

POLI e ZERI della FUNZIONE di TRASFERIMENTO

Data $H(z)$, si dicono:

★ ZERI di $H(z)$: i valori di z per cui $H(z)=0$ ZERI: $z_i \in \mathbb{C}$ t.c. $H(z=z_i)=0$

★ POLI di $H(z)$: i valori di z per cui $H(z)=\pm\infty$ POLI: $p_i \in \mathbb{C}$ t.c. $H(z=p_i)=\pm\infty$

Se $H(z)$ è espressa in forma RAZIONALE:

$$H(z) = \frac{N_m(z^{-1})}{D_n(z^{-1})} = \frac{b_0 + b_1z^{-1} + \dots + b_mz^{-m}}{1 + a_1z^{-1} + \dots + a_nz^{-n}} \quad \begin{array}{l} \longrightarrow \text{ha } m \text{ RADICI} \\ \longrightarrow \text{ha } n \text{ RADICI} \end{array}$$

$$N_m(z^{-1}) = 0 \longrightarrow z = \{z_1, z_2 \dots, z_m\} \longrightarrow H(z=z_i)=0 \rightarrow \{z_i\} : m \text{ ZERI di } H(z)$$

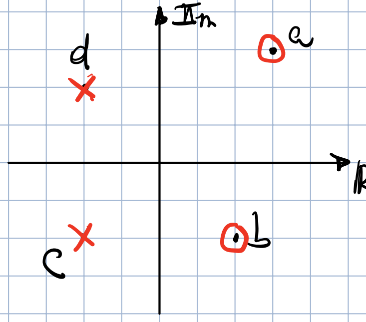
$D_n(z^{-1})=0 \rightarrow z=\{p_1,p_2,\dots p_n\} \rightarrow H(z=p_i)=\pm\infty \rightarrow \{p_i\}$: n POLI di $H(z)$

$$H(z)=\frac{N_m(z^{-1})}{D_n(z^{-1})}=b_0\,\frac{\prod_{1}^{m}{}_i(1-z_iz^{-1})}{\prod_{1}^{n}{}_i(1-p_iz^{-1})}\begin{array}{l}\longrightarrow \{z_i\}:\ m \text{ ZERI di } H(z)\\ \longrightarrow \{p_i\}:\ n \text{ POLI di } H(z)\end{array}$$

DIAGRAMMA POLI - ZERI :

rappr. grafica di POLI e ZERI di $H(z)$ sul PIANO di GAUSS:

- POLI: SIMBOLO: x
- ZERI: SIMBOLO: o



Esempio: $X(z)=\dfrac{(1-az^{-1})(1-bz^{-1})}{(1-cz^{-1})(1-dz^{-1})}\begin{array}{l}\rightarrow \text{ZERI}: \{a,b\}\\ \rightarrow \text{POLI}: \{c,d\}\end{array}$

CAUSALITÀ (nel DOMINIO z)

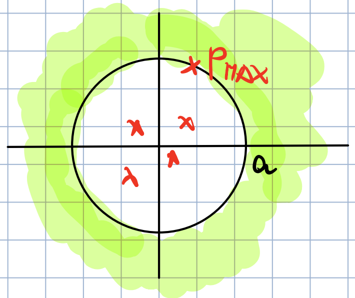
Se un sistema LTI è CAUSALE → $h(n)=0$ per $n<0$

Nel DOMINIO z: considero $H(z)$

$$H(z)=\mathcal{Z}\{h(n)\}=\sum_{-\infty}^{\infty}{}_n h(n)z^{-n}=\left[\begin{array}{c}h(n)=0\\ n<0\end{array}\right]=\sum_{0}^{\infty}{}_n h(n)z^{-n}$$ ← z ha SOLTANTO ESPONENTI NEGATIVI

$H(z)$ è una SERIE di POTENZE con ESP. NEGATIVI

↳ ROC: $|z|>|a|=f(h(n))$



SISTEMA LTI CAUSALE → ROC$[H(z)]$: $|z|>|P_{MAX}|$

P_{MAX} è il POLO con valore di MODULO MASSIMO. dove $P_{MAX}=p_i$ t.c. $MAX\{|p_i|\}$

STABILITÀ (BIBO) nel DOMINIO z

↳ SISTEMA LTI è STABILE sse $\sum_{-\infty}^{\infty}{}_n |h(n)|<\infty$ (val. FINITO)

Nel dominio z:

Considero $|H(z)|_{|z|=1}=\left|\sum_{-\infty}^{\infty}{}_n h(n)z^{-n}\right|\leq\sum_{-\infty}^{\infty}{}_n|h(n)z^{-n}|=\{|z|=1\}=\sum_{-\infty}^{\infty}{}_n|h(n)|\,\underset{\searrow 1}{|z^{-n}|}$ -

$=\sum_{-\infty}^{\infty}{}_n|h(n)|<\infty$ FINITO! → $H(z)$ CONVERGE per $|z|=1$

→ Se sistema LTI è STABILE → H(z) CONVERGE per $|z|=1$ → $\{|z|=1\} \in ROC$

→ " " " è CAUSALE → ROC : $|z| > |p_{max}|$ → $p_{max} = \max\{|p_i|\} < 1$

Un SISTEMA LTI è CAUSALE e STABILE (BIBO) sse TUTTI I POLI p_i di $H(z)$ sono CONTENUTI nel CERCHIO UNITARIO : $|p_i| < 1 \; \forall i$

RISPOSTA in FREQUENZA

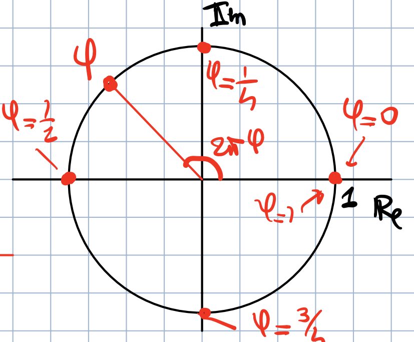
$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

RISPOSTA in FREQUENZA : $DTFT\{h(n)\} = H_{DTFT}(\varphi) = \sum_{-\infty}^{\infty} h(n)\, e^{-j2\pi\varphi n}, \quad 0 \le \varphi < 1$

FUNZ. di TRASFERIMENTO : $H(z) = \mathcal{Z}\{h(n)\} = \sum_{-\infty}^{\infty} h(n)\, z^{-n} \Longrightarrow z = e^{j2\pi\varphi}$

RISPOSTA in FREQUENZA : $H_{DTFT}(\varphi) = H(z)\big|_{\text{con } z = e^{j2\pi\varphi}}, \quad 0 \le \varphi < 1$

Dato $H(z) \longrightarrow H_{DTFT}(\varphi) = H\left(z = e^{j2\pi\varphi}\right), \; 0 \le \varphi < 1$



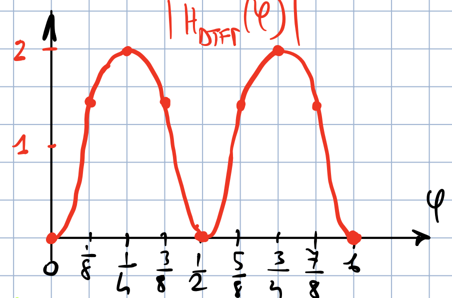
ESEMPIO : risposta in frequenza di : $H(z) = 1 - z^{-2}$

$$H_{DTFT}(\varphi) = H\left(z = e^{j2\pi\varphi}\right) = 1 - e^{-j4\pi\varphi}$$ → RISPOSTA in FREQUENZA

RISPOSTA in AMPIEZZA : $|H_{DTFT}(\varphi)| = \left|1 - e^{-j4\pi\varphi}\right| = \left|\underbrace{1 - \cos(4\pi\varphi)}_{Re} - j \underbrace{\sin(4\pi\varphi)}_{Im}\right| =$

$$= \sqrt{(1-\cos(4\pi\varphi))^2 + \sin^2(4\pi\varphi)} = \sqrt{1 - 2\cos(4\pi\varphi) + \underbrace{\cos^2(4\pi\varphi) + \sin^2(4\pi\varphi)}_{1}} =$$

$$= \sqrt{2 - 2\cos(4\pi\varphi)} = \sqrt{2}\sqrt{1 - \cos(4\pi\varphi)}, \quad 0 \le \varphi < 1$$

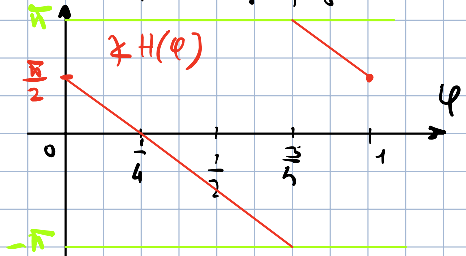


RISPOSTA di FASE :

$$\measuredangle H(\varphi) = \text{atan2} \frac{Im[H(\varphi)]}{Re[H(\varphi)]} = \text{atan}\left[\frac{\sin(4\pi\varphi)}{1 - \cos(4\pi\varphi)}\right] =$$

$$= \text{atan}\left[\text{cotan}(2\pi\varphi)\right] = \left(\text{cotg}\,\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)\right) =$$

$$= \text{atan}\left\{\tan\left(\frac{\pi}{2} - 2\pi\varphi\right)\right\} = \frac{\pi}{2} - 2\pi\varphi$$



ANALISI di SISTEMI LTI DISCRETI (FILTRI NUMERICI)

$$\begin{tikzcd}
x(n) \arrow[r] \arrow[d, "z"'] & \boxed{h(n)} \arrow[r] \arrow[d, "z"'] & y(n) = x(n) * h(n) = \sum_{-\infty}^{\infty} {}_i\, h(i)x(n-i) \arrow[d] \\
X(z) & H(z) & Y(z) = X(z) \cdot H(z)
\end{tikzcd}$$

Consideriamo SISTEMI CAUSALI : $h(n) = 0$ per $n < 0$

$$\longrightarrow \quad y(n) = \sum_{0}^{\infty} {}_i\, h(i)\, x(n-i)$$

✱ SISTEMI con RISPOSTA all'IMPULSO di DURATA FINITA

↳ FILTRI "FIR" : FINITE IMPULSE RESPONSE

Se $h(n)$ è CAUSALE è di DURATA $D+1$ $\longrightarrow$ $h(n) = 0$ per $\begin{cases} n < 0 \\ n > D \end{cases}$

$$y(n) = x(n) * h(n) = \sum_{0}^{D} {}_i\, h(i)\, x(n-i) = \underbrace{h(0)\,x(n)}_{\text{INGRESSO ATTUALE}} + \underbrace{h(1)\,x(n-1) + \dots + h(D)\,x(n-D)}_{\text{D INGRESSI PASSATI}}$$

l'analisi è SEMPLICE

EQUAZIONE alle DIFFERENZE del filtro

Nel DOMINIO Z :

Dato : $h(n) = \{\underset{\uparrow}{h(0)}, h(1), \dots, h(D)\} \xrightarrow{Z} H(z) = \sum_{0}^{D} {}_i\, h(i)\, z^{-i}$ FUNZIONE di TRASFERIMENTO

ANALISI in Z :

Dato : $x(n) \xrightarrow{Z} X(z) \longrightarrow Y(z) = X(z)H(z) \longrightarrow y(n) = Z^{-1}\{Y(z)\}$

ESEMPIO : calcolare l'uscita del filtro con $h(n) = \{\underset{\uparrow}{1}, -1\}$ $\quad h(0) \quad h(1)$

per l'ingresso $x(n) = \{\underset{\uparrow}{1}, 2, 1\}$ $\quad x(0) \quad x(1) \quad x(2)$

Nei TEMPI :

$$y(n) = x(n) * h(n) = \sum_{0}^{1} {}_i\, h(i)\, x(n-i) = h(0)x(n) + h(1)x(n-1) = x(n) - x(n-1)$$

filtro è CAUSALE $\longrightarrow$

USCITA: per

$$n<0 \longrightarrow y(n<0)=0 \quad \text{(CAUSALE)}$$
$$n=0 \longrightarrow y(0)=x(0)-x(-1)=1-0=1$$
$$n=1 \longrightarrow y(1)=x(1)-x(0)=2-1=1$$
$$n=2 \longrightarrow y(2)=x(2)-x(1)=1-2=-1$$
$$n=3 \longrightarrow y(3)=x(3)-x(2)=0-1=-1$$
$$n\geq 4 \longrightarrow y(\geq 4)=x(\geq 4)-x(\geq 3)=0$$

$$\longrightarrow y(n)=\{\underset{\uparrow}{1},1,-1,-1\}$$

Nel DOMINIO Z:

$$x(n)=\{\underset{\uparrow}{1},2,1\} \xrightarrow{Z} X(z)=1z^{0}+2z^{-1}+1z^{-2}=1+2z^{-1}+z^{-2}$$

$$h(n)=\{\underset{\uparrow}{1},-1\} \xrightarrow{Z} H(z)=1-z^{-1}$$

$$\longrightarrow Y(z)=X(z)\cdot H(z)=(1+2z^{-1}+z^{-2})(1-z^{-1})=1+2z^{-1}+z^{-2}-z^{-1}-2z^{-2}-z^{-3}=$$
$$=1+z^{-1}-z^{-2}-z^{-3}$$

$$\longrightarrow y(n)=Z^{-1}\{1+z^{-1}-z^{-2}-z^{-3}\}=\delta(n)+\delta(n-1)-\delta(n-2)-\delta(n-3)$$
$$=\{\underset{\uparrow}{1},1,-1,-1\}$$

★ SISTEMI LTI con RISPOSTA all'IMPULSO di DURATA INFINITA

FILTRI "IIR" - INFINITE IMPULSE RESPONSE

Esempio di IIR: considero $h(n)=u(n)=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases}$

- ANALISI nel dominio dei TEMPI:

$$\longrightarrow y(n)=x(n)*h(n)=\sum_{0}^{\infty}{}_i\, h(i)x(n-i)=\underbrace{\sum_{0}^{\infty}{}_i\, x(n-i)}_{\text{NON CALCOLABILE in PRATICA!}}$$

EQ. alle DIFFERENZE in forma CANONICA

Posso scrivere:

$$y(n)=\sum_{0}^{\infty}{}_i\, x(n-i)=x(n)+\sum_{1}^{\infty}{}_i\, x(n-i)=x(n)+\sum_{0}^{\infty}{}_i\, x(n-(i+1))=$$
$$=x(n)+\underbrace{\sum_{0}^{\infty}{}_i\, x(n-1-i)}_{y(n-1)}=x(n)+y(n-1)$$

Se conosco un VALORE INIZIALE per $y(n)$ (es: $y(n=0) = y_0$)

$$\longrightarrow \quad y(n) = \begin{cases} y_0 & \text{per } n=0 \\ x(n) + y(n-1) & \text{per } n>0 \end{cases}$$

Eq. alle differenze in FORMA RECURSIVA

- Nel DOMINIO Z: la trasformata Z di una sequenza INFINITA è, in genere, l'espressione, nel dominio z, della FORMA RECURSIVA

$$h(n) = u(n) \xrightarrow{Z} H(z) = \frac{1}{1-z^{-1}}$$

L'uscita: $Y(z) = H(z) \cdot X(z) = \frac{X(z)}{1-z^{-1}} \longrightarrow \underbrace{Y(z)(1-z^{-1})}_{\downarrow} = X(z)$

Z^{-1} ← $\boxed{Y(z) - Y(z)z^{-1} = X(z)}$

Antitrasformo: $y(n) - y(n-1) = x(n)$

$$\boxed{y(n) = x(n) + y(n-1)}$$

Eq. diff. in forma RECURSIVA

FILTRI IIR: CASO GENERALE

Filtri caratterizzati da $H(z)$ in FORMA RAZIONALE:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 - a_1 z^{-1} - \dots - a_N z^{-N}} = \frac{\sum_{i=0}^{M} b_i z^{-i}}{1 - \sum_{i=1}^{N} a_i z^{-i}}$$

← parte FIR (INGRESSI)

← parte IIR (USCITE prec.)

Antitrasformo $H(z) = \frac{Y(z)}{X(z)}$, separando $X(z)$ e $Y(z)$:

$$Y(z)\left[1 - \sum_{i=1}^{N} a_i z^{-1}\right] = X(z) \sum_{i=0}^{M} b_i z^{-i}$$

Isolo $Y(z)$: $Y(z) = Y(z) \sum_{i=1}^{N} a_i z^{-i} + X(z) \sum_{i=0}^{M} b_i z^{-i}$

Antitrasformo: $\underset{\substack{\downarrow \\ \text{USCITA ATTUALE}}}{y(n)} = \underbrace{\sum_{i=1}^{N} a_i\, y(n-i)}_{\text{N USCITE PRECEDENTI}} + \underbrace{\sum_{i=0}^{M} b_i\, x(n-i)}_{\text{INGRESSO ATTUALE e M INGRESSI PASSATI}}$

Eq. alle DIFFERENZE RECURSIVA di un FILTRO IIR

Un generico FILTRO IIR si può DESCRIVERE con DUE FORME di EQUAZIONE delle DIFFERENZE:

forma CANONICA: $y(n) = \sum_{i=0}^{\infty} h(i)\, x(n-i)$

forma RECURSIVA: $y(n) = \sum_{1}^{N} {}_i\, a_i\, y(n-i) + \sum_{0}^{M} {}_i\, b_i\, x(n-i)$

STABILITÀ dei FILTRI FIR e IIR

FIR : $h(n) = \{h(0), h(1), \dots, h(M)\} \longrightarrow y(n) = \sum_{0}^{M} {}_i\, h(i)\, x(n-i)$

$$\xrightarrow{Z} H(z) = \sum_{0}^{M} {}_i\, h(i)\, z^{-i} = h_0 + h_1 z^{-1} + \dots + h_M z^{-M} \overset{\text{fattorizzo}}{=} h_0 \prod_{1}^{M} {}_i\, (1 - z_i z^{-1})$$

dove z_i, $i = 1..n$: M ZERI di $H(z)$; NO POLI

→ i filtri FIR sono SEMPRE STABILI

IIR

$$H(z) = \frac{\sum_{0}^{M} {}_i\, b_i\, z^{-i}}{1 - \sum_{1}^{N} {}_i\, a_i z^{-i}} = (\text{fattorizzo}) = \frac{b_0 \prod_{1}^{M} {}_i\, (1 - z_i z^{-1})}{\prod_{1}^{N} {}_i\, (1 - p_i z^{-1})}$$

M ZERI di $H(z)$

N POLI di $H(z)$

→ IIR è STABILE sse $|p_i| < 1, \forall p_i, \; i = 1, \dots, N$

(TUTTI i POLI sono INTERNI alla CIRCONF. UNITARIA)

SINTESI (o PROGETTO) di FILTRI DIGITALI

SINTESI "STANDARD" : IMPLEMENTAZIONE CIRCUITALE dell'EQUAZ. alle DIFFERENZE

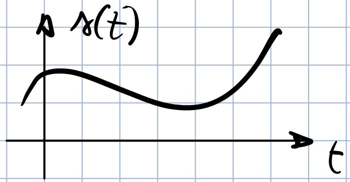
FIR : $y(n) = \sum_{0}^{M} {}_i\, h_i\, x(n-i)$

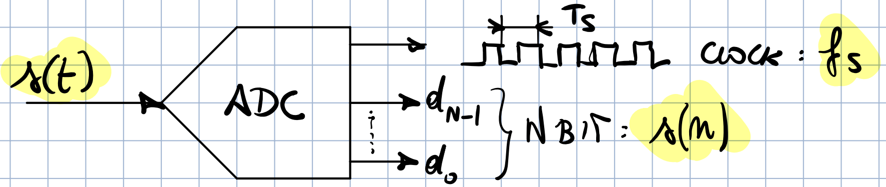
IIR : $y(n) = \sum_{1}^{N} {}_i\, a_i\, y(n-i) + \sum_{0}^{M} {}_i\, b_i\, x(n-i)$ (forma RECURSIVA)

CHE COSA CI SERVE ?

- ★ SEGNALI : $x(n)$
  - → SEGNALE PASSATO : $x(n-1) .. x(n-M)$ $y(n-1) \dots y(n-N)$
- ★ SOMMATORI
- ★ MOLTIPLICATORI

SEGNALI :

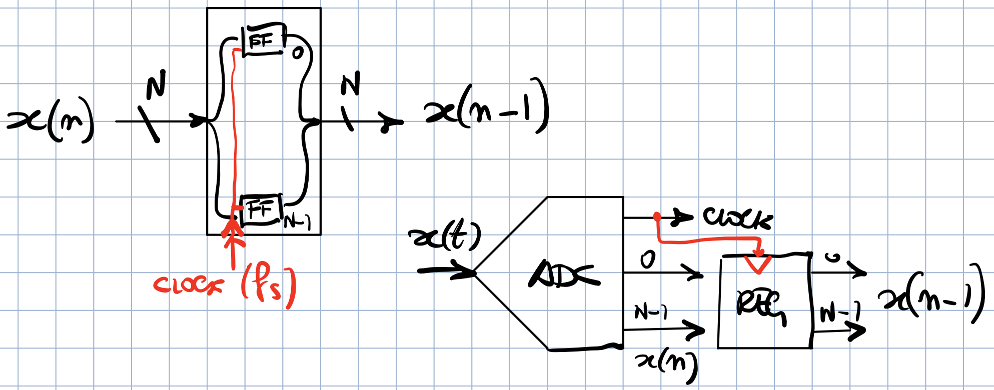




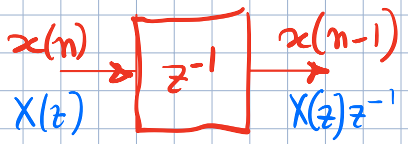
Uscita ADC : flusso digitale di N bit al ritmo di fs campioni/s.

RITARDATORE :

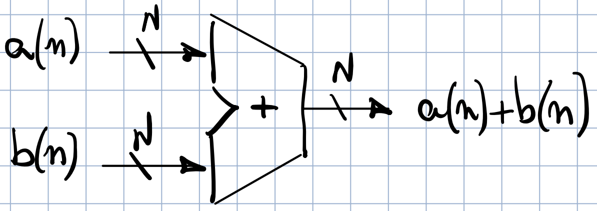
Registro a N bit, comandato dal clock a fs

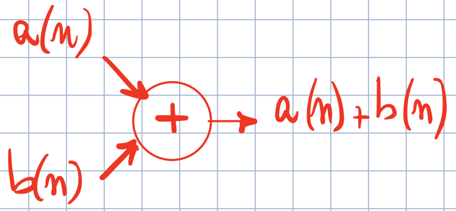


SIMBOLO CIRCUITALE

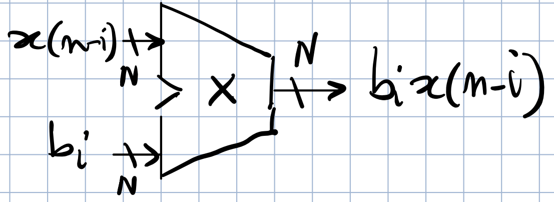


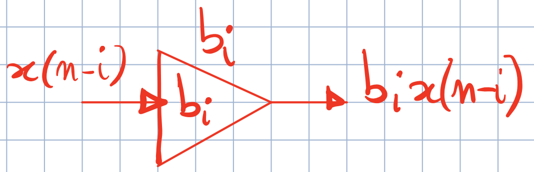
SOMMATORE





MOLTIPLICATORE



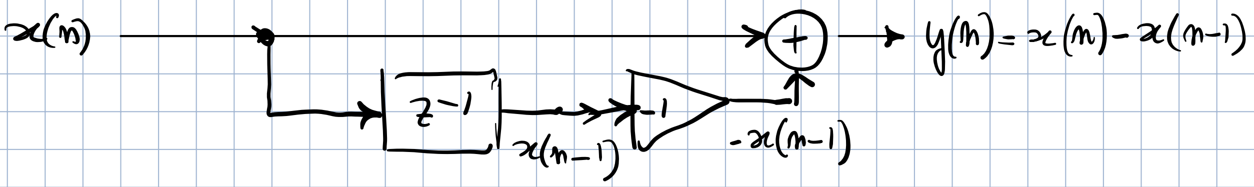


STRUTTURA CIRCUITALE - FILTRI FIR

Esempio : $h(n) = \{1, -1\} \longrightarrow H(z) = 1 - z^{-1}$

$$y(n) = \sum_{0}^{1}{}_i\, h_i x(n-i) = x(n) - x(n-1)$$

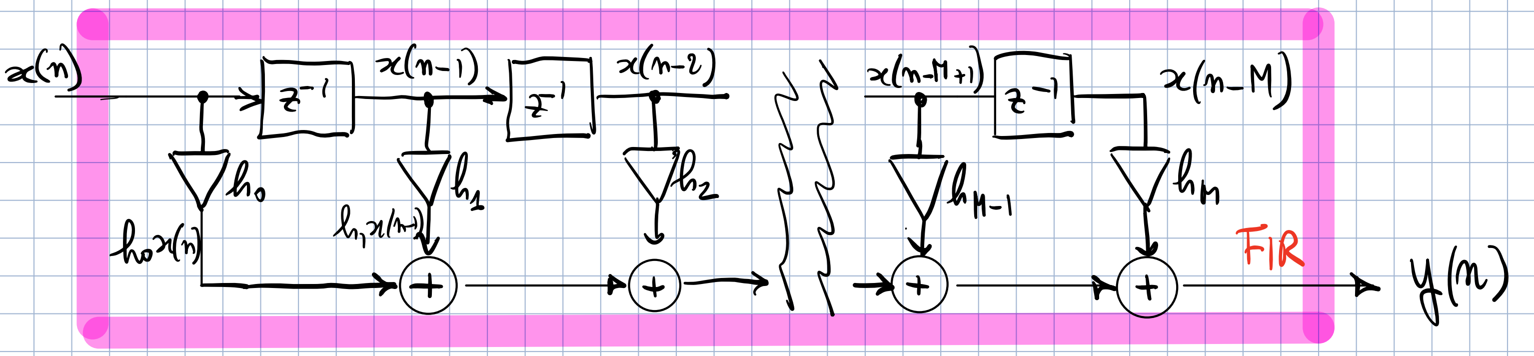
$$Y(z) = \qquad = X(z) - z^{-1} X(z)$$



Per un FIR GENERICO :

$$h(n)=\{\underset{\uparrow}{h_0},h_1,\dots,h_M\} \longrightarrow y(n)=\sum_{i=0}^{M} h_i\, x(n-i) =$$

$$= h_0x(n)+h_1x(n-1)+\dots+h_Mx(n-M)$$



STRUTTURA CIRCUITALE - FILTRI IIR

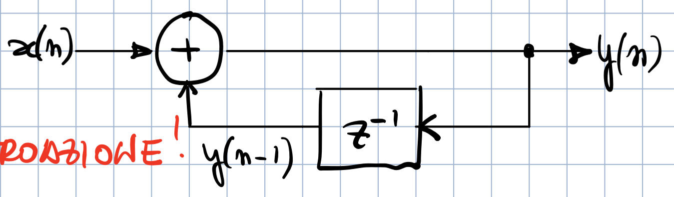
$$y(n)=\sum_{i=1}^{N} a_i\, y(n-i)+\underbrace{\sum_{i=0}^{M} b_i\, x(n-i)}=$$

FIR!

$$=\underbrace{a_1y(n-1)+a_2y(n-2)+\dots+a_Ny(n-N)}+v(n) \quad ; \quad v(n)=\sum_{i=0}^{M} b_i\, x(n-i)$$

Esempio : $y(n)=y(n-1)+x(n)$

↓ forma RECURSIVA

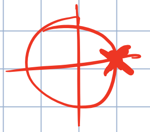


$$Y(z)=Y(z)\,z^{-1}+X(z)$$

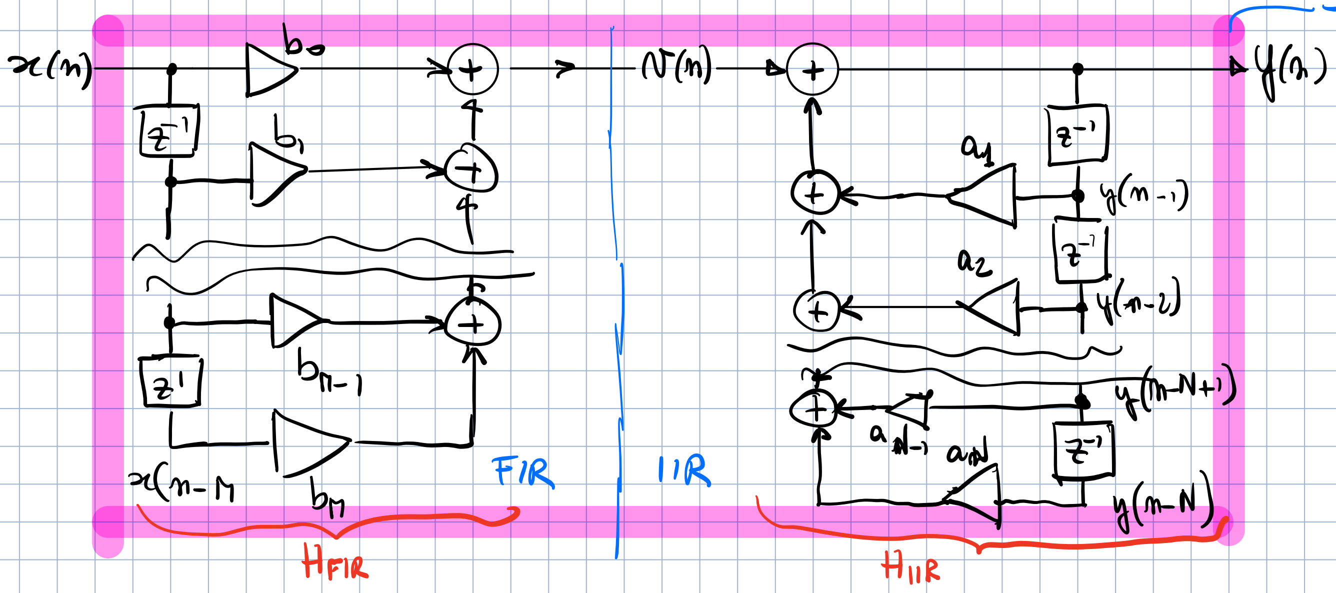
$$Y(z)\,(1-z^{-1})=X(z) \rightarrow H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1-z^{-1}}$$

→ no ZERI

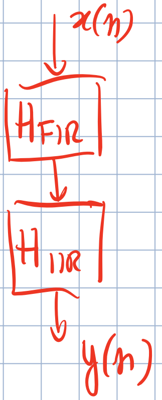
→ 1 POLO : $p=1$



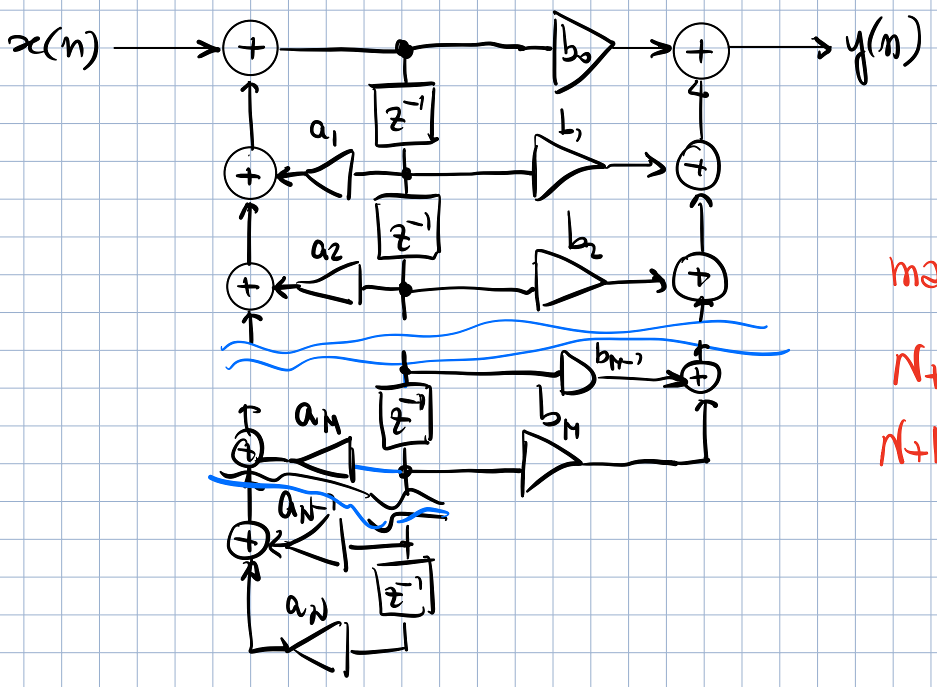
Per un IIR GENERICO : $y(n)=\sum_{i=1}^{N} a_i\, y(n-i)+v(n)$



I FORMA DIRETTA



Invertendo le posizioni di H_{FIR} e H_{IIR} ottengo :



FORMA OTTIMIZZATA (della II f. DIRETTA)

es : $M < N$

$\max(N,M) = N$ RITARDATORI

$N+M$ SOMMATORI

$N+M+1$ MOLTIPLICATORI

ESEMPI di FILTRI DIGITALI

FILTRO "A PETTINE" (COMB FILTER)

Famiglia di filtri FIR

FILTRO a PETTINE di ORDINE M : $\boxed{H(z) = 1 - z^{-M}}$

Dato che : $H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-M} \longrightarrow \underbrace{Y(z) = (1 - z^{-M})X(z) = X(z) - X(z)z^{-M}}$

Z^{-1}

$\boxed{y(n) = x(n) - x(n-M)}$: EQUAZ. alle DIFFERENZE

DIAGRAMMA POLI-ZERI :

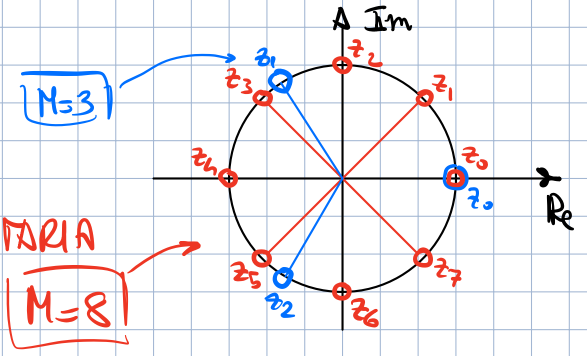
- POLI : NO POLI
- ZERI : M ZERI → le RADICI di : $1 - z^{-M} = 0$

Risolviamo : $1 - z^{-M} = 0 \longrightarrow z^{M} - 1 = 0 \longrightarrow z^{M} = 1 \longrightarrow \boxed{z = \sqrt[M]{1}}$

$z = \sqrt[M]{1} \longrightarrow$ M SOLUZIONI : $\boxed{z_i = \sqrt[M]{1} = e^{j\frac{2\pi}{M}i}, \quad i = 0, \dots, M-1}$

→ M ZERI z_i : $\begin{cases} |z_i| = 1 \\ \measuredangle z_i = \frac{2\pi}{M} i, \quad i=0..M-1 \end{cases}$

M ZERI : M VALORI EQUIDISTANTI sulle CIRCONF. UNITARIA



RISPOSTA in FREQUENZA :

dal diagramma P/Z : $H(\varphi) = 0$ per $\varphi = \frac{i}{M}$, $i = 0, \dots, M-1$

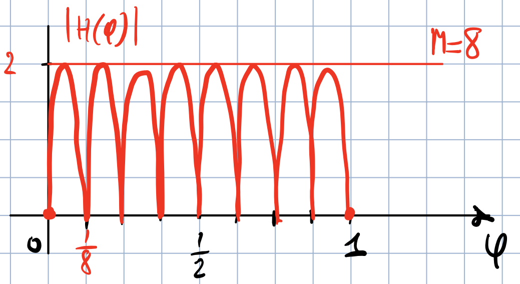
In generale :

$$H_{DIFF}(\varphi) = H(z = e^{j2\pi\varphi}) = 1 - e^{-j2\pi\varphi M} = \underbrace{1 - \cos(2\pi M\varphi)}_{Re} + j\underbrace{\sin(2\pi M\varphi)}_{Im}$$

RISPOSTA in AMPIEZZA : $|H(\varphi)|$:

$$|H(\varphi)| = \sqrt{(1-\cos(2\pi M\varphi))^2 + \sin^2(2\pi M\varphi)} = \sqrt{1 - 2\cos(2\pi M\varphi) + \underbrace{\cos^2(2\pi M\varphi) + \sin^2(2\pi M\varphi)}_{1}} =$$

$$= \sqrt{2 - 2\cos(2\pi M\varphi)} = \sqrt{2}\cdot\sqrt{1-\cos(2\pi M\varphi)}$$

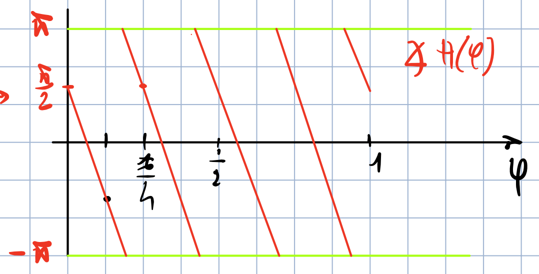


RISPOSTA di FASE : $\measuredangle H(\varphi)$

$$\measuredangle H(\varphi) = \text{atan2}\left[\frac{\sin(2\pi M\varphi)}{1-\cos(2\pi M\varphi)}\right] = \text{atan2}\left[\cot(\pi M\varphi)\right] =$$

$$= \frac{\pi}{2} - M\pi\varphi$$

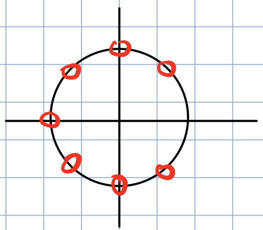
M=8 : →



VARIANTE : CANCELLO lo ZERO per $\varphi = 0$

→ $z = 1$ → con UN POLO in $z = 1$

$$H(z) = \frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}} = \frac{1}{M}\frac{\prod_{i=0}^{M-1}(1-z_iz^{-1})}{1-z^{-1}} = (z_0 = 1) = \frac{1}{M}\prod_{i=1}^{M-1}(1-z_iz^{-1}) \quad \bigstar$$



→ $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}} \rightarrow Y(z)(1-z^{-1}) = \frac{1}{M}(1-z^{-M})X(z)$

Antitrasformo : $y(n) - y(n-1) = \frac{1}{M}[x(n) - x(n-M)]$

$$\boxed{y(n) = y(n-1) + \frac{1}{M}x(n) - \frac{1}{M}x(n-M)}$$ eq. alle DIFFERENZE

$\bigstar$ $H(z) = \frac{1}{M}\prod_{i=1}^{M-1}(1-z_iz^{-1}) = \frac{1}{M}\left[1 + b_1z^{-1} + b_2z^{-2} \dots + b_{M-1}z^{-(M-1)}\right]$

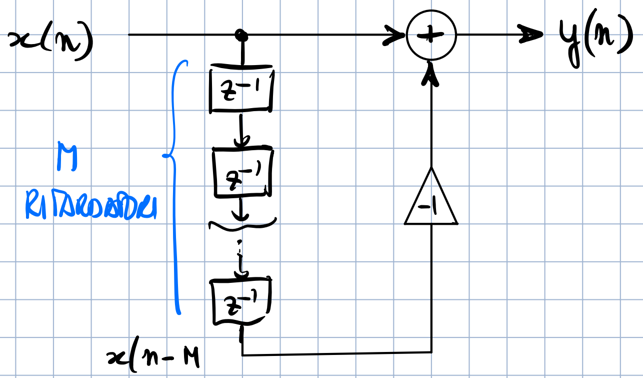
$$z^{-1} \rightarrow h(n) = \frac{1}{M}\left[\delta(n) + b_1\delta(n-1) + \dots + b_{M-1}\delta(n-M+1)\right]$$

$$\rightarrow y(n) = h(n) * x(n) = \frac{1}{M}\left[x(n) + b_1 x(n-1) + \dots b_{M-1} x(n-M+1)\right]$$

STRUTTURA CIRCUITALE

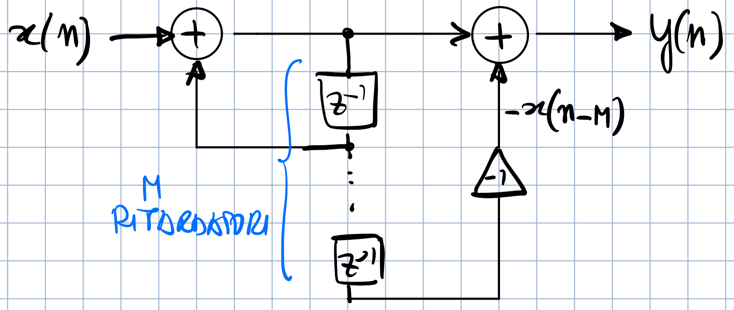
FILTRO a PETTINE, ORDINE M :

$$y(n) = x(n) - x(n-M)$$



ORDINE M, MODIFICATO :

$$y(n) = \underbrace{y(n-1)}_{} + x(n) - x(n-M)$$



FILTRO "NOTCH" (NOTCH FILTER)

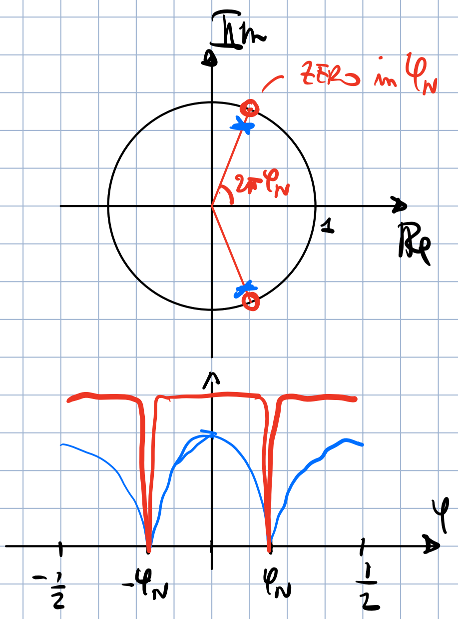
Famiglia di filtri ARRESTA-BANDA

Date φ_N : (freq da arrestare $f_N = \varphi_N \cdot f_s$)

Progettiamo sul DIAGRAMMA POLI/ZERI

2 ZERI a $\pm\varphi_N \rightarrow H(z = e^{\pm j2\pi\varphi_N}) = 0$

$$z_N = e^{j2\pi\varphi_N} \; ; \; \bar{z}_N = e^{-j2\pi\varphi_N}$$



→ 2 POLI, a $\pm\varphi_N$, scalati di α rispetto a z_n, $0 \le \alpha < 1$

$$p_N = \alpha z_N = \alpha e^{j2\pi\varphi_N} \; ; \; \bar{p}_N = \alpha \bar{z}_N \; , \; 0 \le \alpha < 1$$

$$\rightarrow \boxed{H(z) = \frac{(1 - z_N z^{-1})(1 - \bar{z}_N z^{-1})}{(1 - p_N z^{-1})(1 - \bar{p}_N z^{-1})} \; ; \quad \begin{array}{l} z_N = e^{j2\pi\varphi_N} \\ p_N = \alpha z_N = \alpha e^{j2\pi\varphi_N} , \; 0 \le \alpha < 1 \end{array}}$$

$$= \frac{1 - (z_N + \bar{z}_N) z^{-1} + z_N \bar{z}_N z^{-2}}{1 - (p_N + \bar{p}_N) z^{-1} + p_N \bar{p}_N z^{-2}} = \frac{1 - 2\,\mathrm{Re}[z_N] z^{-1} + z^{-2}}{1 - 2\alpha\,\mathrm{Re}[z_N] z^{-1} + \alpha^2 z^{-2}} =$$

$$H(z) = \frac{1 - 2\cos(2\pi\varphi_N)\, z^{-1} + z^{-2}}{1 - 2\alpha\cos(2\pi\varphi_N)\, z^{-1} + \alpha^2 z^{-2}} = \frac{Y(z)}{X(z)}$$

TUTTI i COEFFICIENTI sono REALI !

EQUAZIONE alle DIFFERENZE :

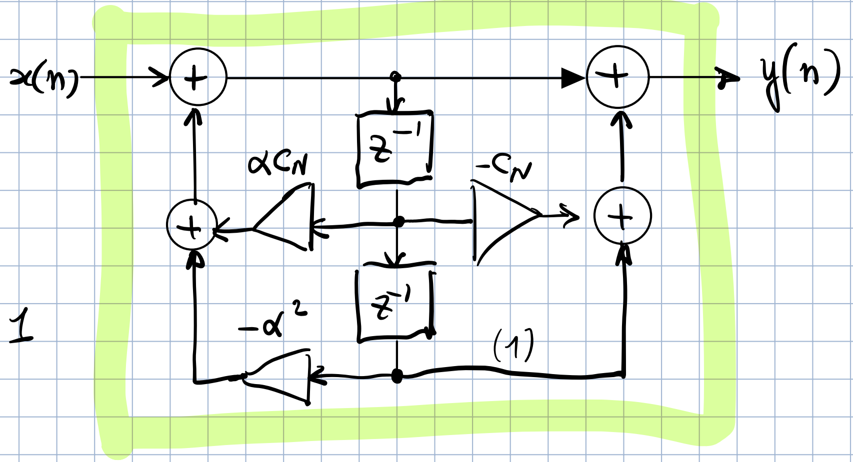
$$Y(z)\left(1 - 2\alpha\cos 2\pi\varphi_N \overset{z^{-1}}{} + \alpha^2 z^{-2}\right) = X(z)\left(1 - 2\cos(2\pi\varphi_N)z^{-1} + z^{-2}\right); \quad c_N = 2\cos(2\pi\varphi_N)$$

$$\Big\downarrow \mathcal{Z}^{-1}$$

$$y(n) - \alpha c_N\, y(n-1) + \alpha^2 y(n-2) = x(n) - c_N\, x(n-1) + x(n-2)$$

$$y(n) = \underbrace{\alpha c_N\, y(n-1) - \alpha^2 y(n-2)}_{IIR} + \underbrace{x(n) - c_N\, x(n-1) + x(n-2)}_{FIR}$$

STRUTTURA CIRCUITALE :



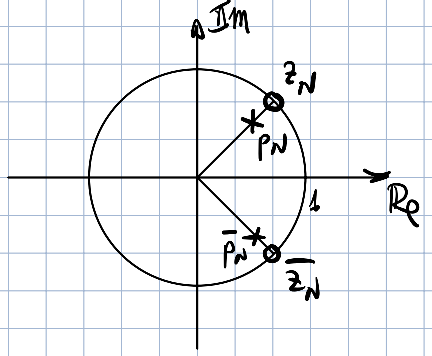
RUOLO di α:

α regola la SELETTIVITA' del FILTRO NOTCH: quanto più $\alpha \to 1$ tanto più il filtro è SELETTIVO

ESERCIZI di RIEPILOGO

PROGETTO filtro NOTCH (Es Tema feb 2023)

$f_N = 50$ Hz $\quad f_s = 400$ Hz $\quad \alpha = 0,8$



a) $\varphi_N = \frac{f_N}{f_s} = \frac{50 \text{ Hz}}{400 \text{ Hz}} = \frac{1}{8}$

$z_N = e^{j2\pi\varphi_N} = e^{j2\pi\frac{1}{8}} = e^{j\frac{\pi}{4}}$; $\bar{z}_N = e^{-j\frac{\pi}{4}}$

$p_N = \alpha z_N = 0,8\, e^{j\pi/4}$ $\quad \bar{p}_N = 0,8\, e^{-j\frac{\pi}{4}}$

$$H(z) = \frac{(1-z_N z^{-1})(1-\bar{z}_N z^{-1})}{(1-p_N z^{-1})(1-\bar{p}_N z^{-1})} = \frac{1-2\cos(\frac{\pi}{4})z^{-1}+z^{-2}}{1-2\alpha\cos(\frac{\pi}{4})z^{-1}+\alpha^2 z^{-2}} = \boxed{\frac{1-\sqrt{2}z^{-1}+z^{-2}}{1-0,8\sqrt{2}z^{-1}+0,64z^{-2}}} = \frac{Y(z)}{X(z)}$$

b) $X(z)(1-\sqrt{2}z^{-1}+z^{-2}) = Y(z)(1-0,8\sqrt{2}z^{-1}+0,64z^{-2})$

$\downarrow \mathcal{Z}^{-1}$

$x(n)-\sqrt{2}x(n-1)+x(n-2) = y(n)-0,8\sqrt{2}y(n-1)+0,64y(n-2)$

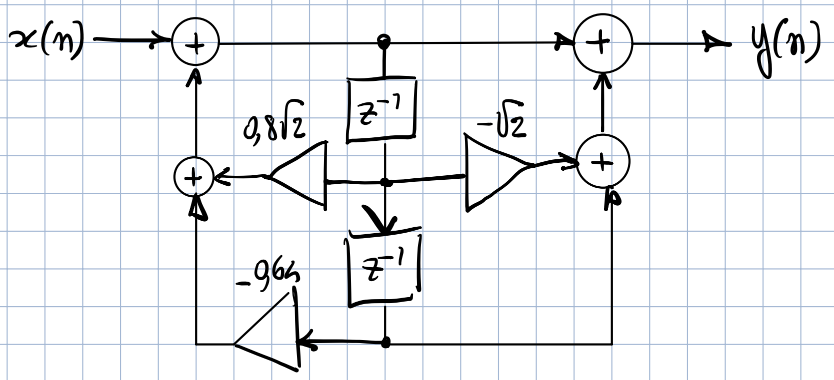
Isolo $y(n)$: $y(n) = 0,8\sqrt{2}y(n-1)-0,64y(n-2)+x(n)-\sqrt{2}x(n-1)+x(n-2)$

c) $H(\varphi)$, $f = 200$ Hz $\longrightarrow \varphi = f/f_s = \frac{200 \text{ Hz}}{400 \text{ Hz}} = \frac{1}{2}$

$$H_{DTFT}(\varphi=\tfrac{1}{2}) = H(z = e^{j2\pi\frac{1}{2}} = e^{j\pi} = -1) = \frac{1-\sqrt{2}(-1)+(-1)^{-2}}{1-0,8\sqrt{2}(-1)+0,64(-1)^2} = \frac{1+\sqrt{2}+1}{1+0,8\sqrt{2}+0,64} =$$

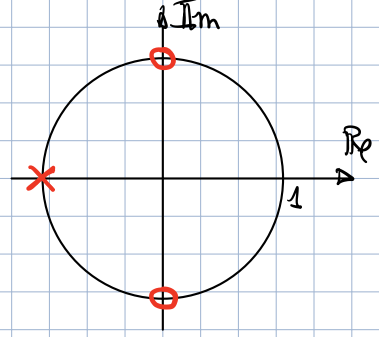
$$= \frac{2+\sqrt{2}}{1,64+0,8\sqrt{2}} \cong 1,23 \rightarrow \begin{cases} |H(\varphi=\frac{1}{2})| = 1,23 \\ \angle H(\varphi=\frac{1}{2}) = 0 \end{cases}$$

d) CIRCUITO :



Tema GIU 2022 :

$f_s = 300\ Hz \qquad f = 75\ Hz \qquad p_1 = -1$



a) $\varphi = \frac{f}{f_s} = \frac{75\ Hz}{300\ Hz} = \frac{1}{4} \rightarrow z_{1,2} = e^{\pm j 2\pi \frac{1}{4}} = e^{\pm j\frac{\pi}{2}} = \pm j$

$$H(z) = \frac{(1-jz^{-1})(1+jz^{-1})}{1+z^{-1}} = \frac{1+z^{-2}}{1+z^{-1}} = \frac{Y(z)}{X(z)}$$

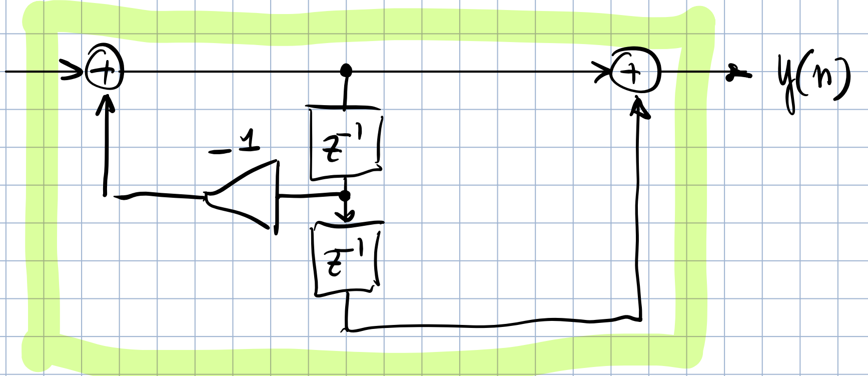
b) $X(z) + X(z)z^{-2} = Y(z) + Y(z)z^{-1}$

$\downarrow \mathcal{Z}^{-1}$

$x(n) + x(n-2) = y(n) + y(n-1) \longrightarrow y(n) = -y(n-1) + x(n) + x(n-2)$

c) $H_{DTFT}(\varphi=0) = H(z = e^{j2\pi 0} = 1) = \frac{1+1}{1+1} = 1 \longrightarrow \begin{cases} |H(0)| = 1 \\ \measuredangle H(0) = 0 \end{cases}$

d) CIRCUITO :



e) $h(n) = \mathcal{Z}^{-1}\{H(z)\}$

$a^n u(n) \longrightarrow \frac{1}{1-az^{-1}}$

$$H(z) = \frac{1+z^{-2}}{1+z^{-1}} = \frac{1}{1+z^{-1}} + \frac{z^{-2}}{1+z^{-1}}$$

$\mathcal{Z}^{-1} \downarrow$

$h(n) = (-1)^n u(n) + (-1)^{n-2} u(n-2) = \delta(n) - \delta(n-1) + 2(-1)^{n-2} u(n-2)$

$(-1)^n u(n) \rightarrow \{\underset{\uparrow}{1}, -1, 1, -1, \dots\}$

$(-1)^{n-2} u(n-2) \hookrightarrow \{\underset{\uparrow}{0}, 0, 1, -1, 1, -1\}$

$h(n) = \{\underset{\uparrow}{1}, -1, 2, -2, 2, -2, \dots\}$