TRASFORMATE per SEGNALI DIGITALI

TRASFORMATA di FOURIER per SEGNALI TEMPO-DISCRETI

Dato $s(t)$ TEMPO-CONTINUO $\longrightarrow$ $s_c(t)$ CAMPIONATO (TEMPO-DISCRETO)

$$s(t) \overset{\mathcal{F}}{\longleftrightarrow} S(f)$$

$$s_c(t) = \sum_{-\infty}^{\infty} s(nT_s)\,\delta(t - nT_s) \longleftrightarrow S_c(f) = \frac{1}{T_s}\sum_{-\infty}^{\infty} S\left(f - \frac{n}{T_s}\right)$$

PERIODICO "PERIODO" $\frac{1}{T} = f_s$

Riusciamo ad esprimere $S_c(f)$ in FUNZIONE solo dei CAMPIONI $s(nT_s)$

$$s_c(t) = \sum_{-\infty}^{\infty} s(nT_s)\,\delta(t - nT_s)$$

Calcolo lo spettro di $s_c(t)$ partendo da :

$$s_c(t) \longrightarrow S_c(f) = \int_{-\infty}^{\infty} s_c(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} \sum_{-\infty}^{\infty} s(nT_s)\,\delta(t - nT_s)\, e^{-j2\pi f t}\, dt =$$

$$= (\times \text{ linearità}) : = \sum_{-\infty}^{\infty} s(nT_s) \underbrace{\int_{-\infty}^{\infty} \delta(t - nT_s)\, e^{-j2\pi f t}\, dt}_{= e^{-j2\pi f n T_s}} = \sum_{-\infty}^{\infty} s(nT_s)\, e^{-j2\pi f n T_s}$$

$$\boxed{s_c(t) \overset{\mathcal{F}}{\longleftrightarrow} S_c(f) = \sum_{-\infty}^{\infty} s(nT_s)\, e^{-j2\pi f n T_s}}$$

$\{s_n = s(nT_s)\}$

Verifico che $S_c(f)$ sia PERIODICA con PERIODO $f_s = \frac{1}{T_s}$:

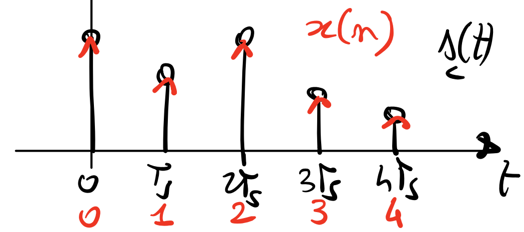
$f_s T_s = 1$

$$S_c(f + k f_s) = \sum_{-\infty}^{\infty} s(nT_s)\, e^{-j2\pi (f + k f_s) n T_s} = \underbrace{\sum_{-\infty}^{\infty} s(nT_s)\, e^{-j2\pi f n T_s}}_{S_c(f)}\, \underbrace{e^{-j2\pi k f_s n T_s}}_{1} =$$

$$= S(f) \;\checkmark$$

Cerchiamo una versione "NORMALIZZATA" di $S_c(f)$ (rispetto a T_s / f_s)

CONSIDERO : $x(n) = s_c(nT_s)$



$$x(n) \longrightarrow S_c(f) = \sum_{-\infty}^{\infty} x(n)\, e^{-j2\pi f n T_s}$$

DEFINISCO la FREQUENZA NORMALIZZATA:

$$\boxed{\varphi = f/f_s = f \cdot T_s}$$

$$\longrightarrow \boxed{S_c(f) = \sum_{-\infty}^{\infty} x(n)\, e^{-j2\pi n \varphi} = X(\varphi), \quad 0 \le \varphi < 1 \quad \text{PERIODICO PERIODO } \varphi = 1}$$

$$\boxed{x(n),\ n \in \mathbb{Z} \xleftrightarrow{\mathcal{F}} X(\varphi) = \sum_{-\infty}^{\infty} x(n)\, e^{-j2\pi n \varphi}, \quad 0 \le \varphi < 1}$$

$X(\varphi)$: TRASFORMATA di FOURIER a TEMPO DISCRETO /
DISCRETE-TIME FOURIER TRANSFORM: DTFT

$X(\varphi)$ è PERIODICA (PERIODO φ: 1) $\longrightarrow$ SERIE di FOURIER

FORMULA di SINTESI $\longrightarrow$ $X(\varphi) = \sum c_k\, e^{j2\pi \frac{n}{1}\varphi} \longrightarrow \begin{cases} x(n) = c_k \\ \text{FASORI INVERTITI} \end{cases}$

$\Longrightarrow$ $DTFT^{-1}$: formula di ANALISI con: $\begin{cases} \text{FASORI INVERTITI} \\ c_k \Leftrightarrow x(n) \end{cases}$

$$\{c_k =\}\ x(n) = \frac{1}{T}\int_T X(\varphi)\, e^{j2\pi \frac{n}{T}\varphi}\, d\varphi = (T=1) = \int_0^1 X(\varphi)\, e^{j2\pi n \varphi}\, d\varphi$$

TRASFORMATA di FOURIER a TEMPO DISCRETO: DTFT

Formula di ANALISI: $x(n),\ n \in \mathbb{Z} \xrightarrow{DTFT} X(\varphi) = \sum_{-\infty}^{\infty} x(n)\, e^{-j2\pi n \varphi}$

Formula di SINTESI: $X(\varphi),\ \varphi \in \mathbb{R} \xrightarrow{DTFT^{-1}} x(n) = \int_0^1 X(\varphi)\, e^{j2\pi n \varphi}\, d\varphi$

PROPRIETÀ della DTFT

TRASLAZIONE nei TEMPI (DISCRETI)

$$x(n) \longleftrightarrow X(\varphi) \implies x(n-m) \longleftrightarrow X(\varphi)e^{-j2\pi\varphi m}$$

DIM: $x(n-m) \xrightarrow{DTFT} X'(\varphi) = \sum_{-\infty}^{\infty} x(n-m)\, e^{-j2\pi n\varphi} = (n' = n-m) =$

$$= \sum_{-\infty}^{\infty} x(n')\, e^{-j2\pi(n'+m)\varphi} = \underbrace{\sum_{-\infty}^{\infty} x(n')\, e^{-j2\pi n'\varphi}}_{X(\varphi)} \cdot \underbrace{e^{-j2\pi m\varphi}}_{\text{COSTANTE}} = X(\varphi)e^{-j2\pi\varphi m} \quad \checkmark$$

TRASLAZIONE nelle FREQUENZE

$$x(n) \xleftrightarrow{DTFT} X(\varphi) \longrightarrow x(n)e^{j2\pi n\varphi_0} \xleftrightarrow{DTFT} X(\varphi-\varphi_0)$$

CONVOLUZIONE

$$\begin{array}{l} x(n) \longleftrightarrow X(\varphi) \\ y(n) \longleftrightarrow Y(\varphi) \end{array} \implies x(n) * y(n) \xleftrightarrow{DTFT} X(\varphi)\cdot Y(\varphi)$$

$s(n)$ DISCRETO $\xrightarrow{DTFT}$ $S(\varphi)$ è PERIODICO

$s(n)$ PERIODICO $\xrightarrow{FS}$ $\{c_k\}$ è DISCRETO

$x(n)$ DISCRETO e PERIODICO $\longrightarrow$ $X(\varphi_k)$ è PERIODICO e DISCRETO $\rightarrow$ $\varphi \in \mathbb{R}$, φ_k, $k \in \mathbb{Z}$

TRASFORMATA di un SEGNALE DIGITALE

Segnali digitali: SEQUENZA di N CAMPIONI

Sia dato $s(n)$ di N CAMPIONI: $n = 0, 1, \dots, N-1$

Ne definiamo una versione PERIODICIZZATA: $x(n)$:

$s(n)$, $n = 0, \dots, N-1 \longrightarrow x(n) : x(n+kN) = s(n)$, $\forall k \in \mathbb{Z}$

↳ DISCRETO e PERIODICO!

$x(n)$ è DISCRETO $\xrightarrow{DTFT}$ $X(\varphi) = \sum_{-\infty}^{\infty} x(n) e^{-j2\pi n\varphi}$, $0 \le \varphi < 1$

$x(n)$ è PERIODICO $\xrightarrow{FS}$ $x(n) = \sum_{-\infty}^{\infty} c_k e^{j2\pi \frac{k}{N} n}$ $\{c_k\}$ è PERIODICA

Considero: $x(n) = \sum_{0}^{N-1} c_k e^{j2\pi \frac{k}{N} n}$, $n = 0, 1, \dots, N-1$

Posso esprimere $\{c_k\}$ mediante la formula di ANALISI (di FS):

$c_k = \frac{1}{T} \int_T x(t) e^{-j2\pi \frac{k}{T} t} dt$; nel nostro caso: $\begin{cases} \star\ x(t) \to x(n) \text{ DISCRETO} \\ \star\ T \to N \text{ (PERIODO)} \end{cases}$

Quindi: $\boxed{c_k = \frac{1}{N} \sum_{0}^{N-1} x(n) e^{-j2\pi \frac{k}{N} n}}$, $0 \le \varphi = \frac{k}{N} < 1 \to k = 0, 1, \dots, N-1$

$0 \le \varphi < 1$

$= \frac{1}{N} X'(\varphi = \frac{k}{N})$

Definisco: $\sum_{0}^{N-1} x(n) e^{-j2\pi \frac{k}{N} n} = Nc_k = X'(\varphi = \frac{k}{N}) = X(k) = \text{DFT}\{x(n)\}$

DISCRETE FOURIER TRANSFORM: DFT:

Dato un SEGNALE DISCRETO di N CAMPIONI: $x(n)$, $n = 0, \dots, N-1$ (considerato PERIODICIZZATO con PERIODO = N):

$$x(n) \xrightarrow{DFT} X(k) = \sum_{0}^{N-1} x(n) e^{-j2\pi \frac{k}{N} n} \qquad n, k = 0, \dots, N-1$$

e la DFT INVERSA (IDFT, DFT^{-1}):

$$X(k) \xrightarrow{IDFT} x(n) = \frac{1}{N} \sum_{0}^{N-1} X(k) e^{j2\pi \frac{k}{N} n}, \qquad n, k = 0, \dots, N-1$$

$\star$ DFT: RELAZIONE BIUNIVOCA tra SEQUENZE di N CAMPIONI

★ La DFT CORRISPONDE alla DTFT del segnale PERIODICIZZATO

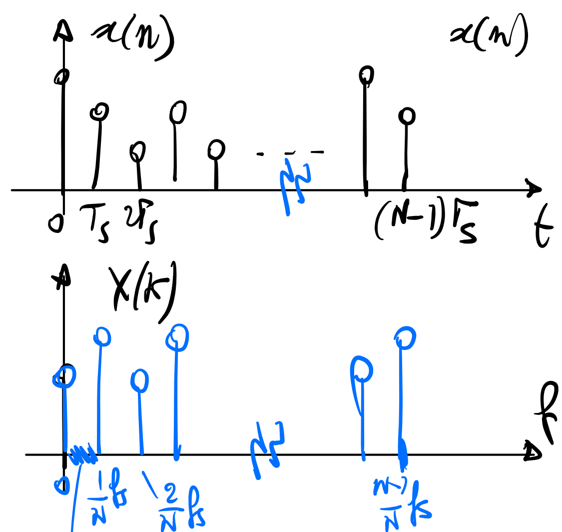
$$x(n) \xleftrightarrow{DFT} X(k) \qquad n,k = 0,\dots,N-1$$

$$x(n) \text{ PERIODICIZZATO} \xleftrightarrow{DTFT} X(k) \text{ PERIODICIZZATA} \qquad n,k \in \mathbb{Z}$$

RISOLUZIONE e DURATA nella DFT :

Considero un segnale di N CAMPIONI, CAMPIONATO ogni T_s $(f_s = 1/T_s)$

	RISOLUZIONE	DURATA
nei TEMPI	T_s	$N T_s$
nelle FREQUENZE	$\Delta f = \frac{f_s}{N} = \frac{1}{N T_s}$	$f_s = \frac{1}{T_s}$



$$\Delta f = \frac{f_s}{N} = \frac{1}{N T_s}$$

ESEMPIO: segnale campionato a $f_s = 48\,kHz$

→ voglio una DFT con PASSO di $10\,Hz = \Delta f$

COME FARE la DFT ?

$$\Delta f = 10\,Hz = \frac{1}{N T_s} = \frac{f_s}{N} \longrightarrow N = \frac{f_s}{10\,Hz} = 4800 \text{ campioni}$$

$$\text{DURATA} = N T_s = \frac{1}{10} = 0{,}1\,s$$

CALCOLO della DFT in FORMA MATRICIALE

Date le espressioni della DFT :

$$\begin{cases} X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi \frac{nk}{N}} \\ x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi \frac{nk}{N}} \end{cases} \qquad n,k = 0,\dots,N-1$$

Posso esprimere DFT e IDFT come PRODOTTO MATRICE-VETTORE :

$$\text{DFT}: \begin{bmatrix} X(0)\\ X(1)\\ \vdots \\ X(k)\\ \vdots\\ X(N-1)\end{bmatrix} = \overbrace{\begin{bmatrix} W_{ij} = e^{-j2\pi\frac{ij}{N}} \\ \\ i,j=0,\dots,N-1\end{bmatrix}}^{F} \begin{bmatrix} x(0)\\ x(1)\\ \vdots\\ x(n)\\ \vdots\\ x(N-1)\end{bmatrix} \longrightarrow \underline{X} = F\cdot \underline{x}$$

$$\text{IDFT}: \begin{bmatrix} x(0)\\ \\ x(n)\\ \\ x(N-1)\end{bmatrix} = \frac{1}{N} \underbrace{\begin{bmatrix} W_{ij} = e^{j2\pi\frac{ij}{N}} \\ \\ i,j=0,\dots,N-1\end{bmatrix}}_{F^{-1}} \begin{bmatrix} X(0)\\ \vdots\\ X(k)\\ \vdots\\ X(N-1)\end{bmatrix} \longrightarrow \underline{x} = F^{-1}\cdot \underline{X} \qquad F^{-1} = \overline{F}\cdot\frac{1}{N}$$

COMPLESSITÀ di CALCOLO della DFT: $O(N^2)$

Esiste un ALGORITMO di calcolo OTTIMIZZATO:

FAST FOURIER TRANSFORM: FFT: COMPLESSITÀ $O(N\log_2 N)$

ESEMPIO: calcolo la DFT con $N=4$ in forma MATRICIALE

$\rightarrow$ devo calcolare $F_{4\times4}$:

$$F_{4\times4} = \begin{bmatrix} W_{nk} = e^{-j2\pi\frac{nk}{N}} \\ n,k=0,\dots,N-1\end{bmatrix} = \begin{bmatrix} W_{nk} = e^{-j\frac{\pi}{2}nk} \\ n,k=0,\dots,3\end{bmatrix} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0\\1\\2\\3 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{\pi}{2}} & e^{-j\pi} & e^{-j\frac{3}{2}\pi} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} \\ 1 & e^{-j\frac{3}{2}\pi} & e^{-j3\pi} & e^{-j\frac{9}{2}\pi} \end{bmatrix} \end{array} =$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$$

Calcoliamo la DFT del segnale $x(n) = \{1, 1, 0, 0\}$

$$X(k) = F\cdot x(n) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1\\1\\0\\0 \end{bmatrix} = \begin{bmatrix} 1+1+0+0 = 2 \\ 1-j \\ 1-1 = 0 \\ 1+j \end{bmatrix} = \begin{bmatrix} 2\\ 1-j \\ 0 \\ 1+j \end{bmatrix} = X(k)$$

IDFT: calcolo $F^{-1} = \frac{1}{N}\overline{F} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}$

$$x(n) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}\begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2+1\cancel{-j}+0+1\cancel{+j} = 4 \\ 2\cancel{+j}+1+0\cancel{-j}+1 = 4 \\ 2-1\cancel{+j}+0-1\cancel{-j} = 0 \\ 2\cancel{-j}-1+0\cancel{+j}-1 = 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

PROPRIETÀ delle DFT:

TRASLAZIONE CICLICA nei TEMPI

$$x(n) \xleftrightarrow{DFT} X(k) \implies x(\langle n-a\rangle_N) \xleftrightarrow{DFT} X(k)\,e^{-j2\pi\frac{a}{N}k}$$

TRASLAZIONE CICLICA nelle FREQUENZE

$$x(n) \xleftrightarrow{DFT} X(k) \implies x(n)\,e^{j2\pi\frac{h}{N}n} \xleftrightarrow{DFT} X(\langle k-h\rangle_N)$$

CONVOLUZIONE DISCRETA CICLICA

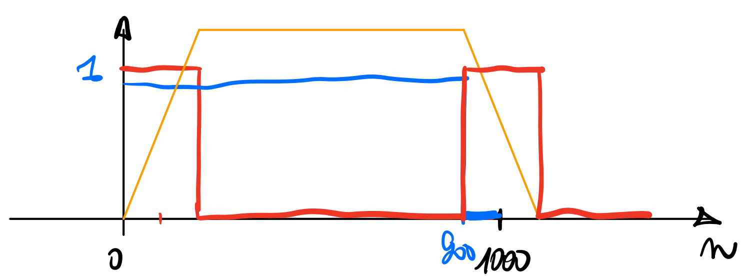
DEF: $x(n) \circledast y(n) = \sum_{i=0}^{N-1} x(i)\,y(\langle n-i\rangle_N) \qquad n = 0, \dots, N-1$

$$\begin{matrix} x(n) \xleftrightarrow{DFT} X(k) \\ y(n) \xleftrightarrow{DFT} Y(k) \end{matrix} \implies x(n) \circledast y(n) \xleftrightarrow{DFT} X(k)\cdot Y(k)$$

ESEMPIO:
$N = 1000$

$x(n) = \mathrm{rect}\left(\frac{n-100}{200}\right)$

$y(n)\ \mathrm{rect}\left(\frac{n-450}{900}\right)$

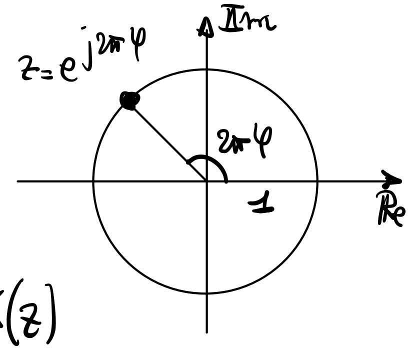


TRASFORMATA Z

Dato un segnale $x(n)$, $n \in \mathbb{Z}$:

$$x(n) \xrightarrow{\text{DTFT}} X(\varphi) = \sum_{-\infty}^{\infty} x(n)\, e^{-j2\pi n\varphi}, \quad 0 \le \varphi < 1$$

Considero: $e^{-j2\pi\varphi n} = f(n) = (\underbrace{e^{j2\pi\varphi}}_{z\in\mathbb{C}})^{-n} = z^{-n}$



Ponendo: $z = e^{j2\pi\varphi}$

$$x(n) \longrightarrow X(\varphi) = \sum_{-\infty}^{\infty} x(n)\, e^{-j2\pi\varphi n} = \sum_{-\infty}^{\infty} x(n)\, z^{-n} = X(z)$$

TRASFORMATA Z di $x(n)$

TRASFORMATA Z:

Dato un SEGNALE DISCRETO $x(n)$, $n\in\mathbb{Z}$, si definisce TRASFORMATA Z di $x(n)$ la funzione della VARIABILE COMPLESSA z:

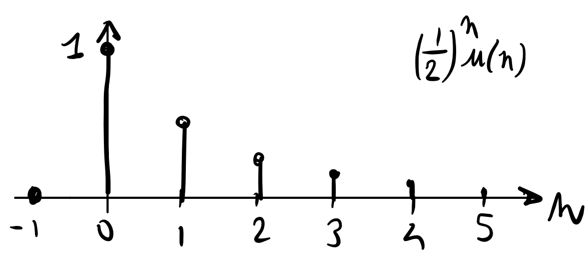
$$\boxed{x(n) \overset{z}{\longleftrightarrow} X(z) = \sum_{-\infty}^{\infty} x(n)\, z^{-n}, \quad n\in\mathbb{Z},\ z\in\mathbb{C}}$$

Il LUOGO dei PUNTI $z\in\mathbb{C}$ per i quali $X(z)$ CONVERGE è detto REGIONE di CONVERGENZA (ROC) di $X(z)$.

Affinché la TRASFORMATA $X(z)$ sia UNIVOCAMENTE DETERMINATA è necessario definire:
- la FUNZIONE $X(z)$
- la sua REGIONE di CONVERGENZA (ROC)

Esempio di calcolo:

$$x(n) = a^n u(n) = \begin{cases} a^n & n \ge 0 \\ 0 & n < 0 \end{cases}$$



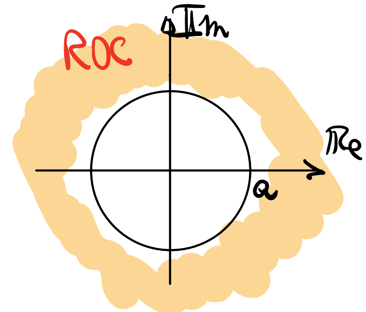
$$X(z) = \sum_{-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{0}^{\infty} a^n z^{-n} = \sum_{0}^{\infty} \left(\frac{a}{z}\right)^n$$

SERIE di POTENZE: $\sum_{0}^{\infty} \alpha^n \longrightarrow$ CONVERGE per $|\alpha| < 1$: $\sum_{0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$

Quindi:

se $\left|\frac{a}{z}\right| < 1 \longrightarrow X(z) = \sum_{0}^{\infty} \left(\frac{a}{z}\right)^n$ CONVERGE: $X(z) = \frac{1}{1-\frac{a}{z}} = \frac{1}{1-az^{-1}}$

Se $\left|\frac{a}{z}\right| \geq 1 \longrightarrow X(z) = \sum_{0}^{\infty} n \left(\frac{a}{z}\right)^n$ DIVERGE : $\nexists X(z)$



ROC : $\left|\frac{a}{z}\right| < 1 \longrightarrow \boxed{|z| > |a|}$

$$x(n) = a^n u(n) \xleftrightarrow{Z} X(z) = \frac{1}{1-az^{-1}} \; ; \; \text{ROC}: |z| > |a|$$

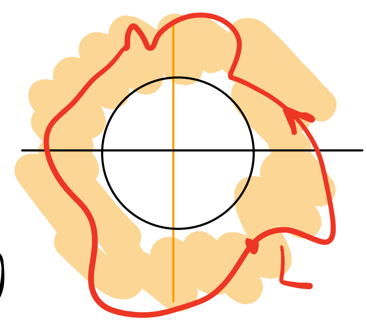
ANTITRASFORMATA Z

Data $X(z)$ trasform. Z di una sequenza $x(n)$, si ottiene $x(n)$ a partire da $X(z)$ con :

$$X(z) \xrightarrow{Z^{-1}} x(n) = \frac{1}{j2\pi} \oint_L X(z) z^{n-1} dz, \quad n \in \mathbb{Z}, \; z \in \mathbb{C}, \; L \in \text{ROC}[X(z)]$$

dove L è un QUALUNQUE PERCORSO CHIUSO che contiene l'ORIGINE, è INTERAMENTE CONTENUTO in ROC[X(z)] e percorso in senso ANTIORARIO

PROPRIETÀ della TRASFORMATA Z



LINEARITÀ

$$\begin{array}{l} x(n) \xleftrightarrow{Z} X(z) \; ; \; \text{Roc}[X] \\ y(n) \xleftrightarrow{Z} Y(z) \; ; \; \text{Roc}[Y] \end{array} \longrightarrow a x(n) + b y(n) \xleftrightarrow{Z} a X(z) + b Y(z)$$

$$\text{Roc}: \text{Roc}[X] \wedge \text{Roc}[Y]$$

TRASLAZIONE nei TEMPI

$$x(n) \xleftrightarrow{Z} X(z) \; ; \; \text{Roc}[X] \longrightarrow x(n-a) \xleftrightarrow{Z} X(z) z^{-a} \quad \text{Roc}: \text{Roc}[X], \begin{cases} z \neq 0 & a > 0 \\ z \neq \pm\infty & a < 0 \end{cases}$$

CONVOLUZIONE

CONV. LINEARE

$$\begin{array}{l} x(n) \leftrightarrow X(z) \\ y(n) \leftrightarrow Y(z) \end{array} \longrightarrow x(n) * y(n) \xleftrightarrow{Z} X(z) \cdot Y(z) \quad \text{ROC}: \text{ROC}[X] \wedge \text{ROC}[Y]$$

COPPIE NOTEVOLI

IMPULSO DISCRETO

$$x(n) = \delta(n) \overset{z}{\longleftrightarrow} X(z) = \sum_{-\infty}^{\infty} {}_n \delta(n) z^{-n} = 1 \quad \text{ROC: } z \in \mathbb{C}$$

IMPULSO TRASLATO

$$x(n) = \delta(n-m) \overset{z}{\longleftrightarrow} X(z) = \sum_{-\infty}^{\infty} {}_n \delta(n-m) z^{-n} = z^{-m} \quad z \in \mathbb{C}, \begin{array}{l} z \neq 0 \;\; m > 0 \\ z \neq \pm\infty \;\; m < 0 \end{array}$$

SEQUENZA ESPONENZIALE

$$x(n) = a^n u(n) \overset{z}{\longleftrightarrow} X(z) = \sum_{0}^{\infty} {}_n a^n z^{-n} = \frac{1}{1-az^{-1}} \; ; \quad \text{ROC: } |z| > |a|$$

GRADINO UNITARIO

$$x(n) = u(n) \overset{z}{\longleftrightarrow} X(z) = \sum_{0}^{\infty} {}_n z^{-n} = \frac{1}{1-z^{-1}} \; ; \quad \text{Roc } |z| > 1$$

Proprietà della Trasformata Z :

DERIVAZIONE

$$x(n) \overset{z}{\longleftrightarrow} X(z) \quad \Longrightarrow \quad n \cdot x(n) \overset{z}{\longleftrightarrow} -z \frac{dX(z)}{dz} \quad \text{ROC: Roc}[X(z)]$$

CALCOLO delle TRASFORMATA Z : casi COMUNI

★ Sequenze ESPONENZIALI : $x(n) = a^n u(n) \overset{z}{\longleftrightarrow} X(z) = \dfrac{1}{1-az^{-1}}$

ROC: $|z| > |a|$

Per LINEARITÀ :

$$x(n) = \sum_i c_i a_i^n u(n) \overset{z}{\longleftrightarrow} X(z) = \sum_i \frac{c_i}{1-a_i z^{-1}}$$

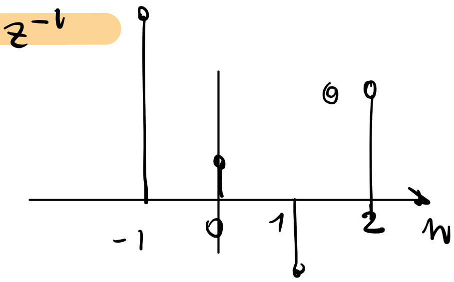
★ Sequenze di LUNGHEZZA FINITA :

$$x(n) = \begin{cases} a_n, & n_0 \leq n \leq n_1 \\ 0 & \text{altrove} \end{cases} \; ; \quad x(n) = \{a_{n_0}, a_{n_0+1}, \dots, a_{n_1}\}$$

$$x(n) = \{a_{n_0}, a_{n_0+1}, \dots, a_{n_1}\} = \sum_{n_0}^{n_1} {}_i \, a_i \, \delta(n-i)$$

$$x(n)=\sum_{m_0}^{m_1} a_i\,\delta(n-i) \xleftrightarrow{Z} Z\left\{\sum_{m_0}^{m_1} a_i\,\delta(n-i)\right\} \overset{\text{LINEARITÀ}}{=} \sum_{m_0}^{m_1} a_i\, Z\{\delta(n-i)\} = \sum_{m_0}^{m_1} a_i z^{-i}$$

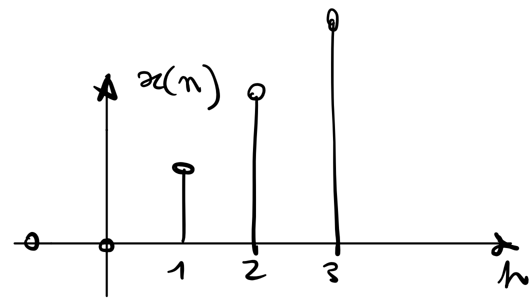
$$x(n)=\{a_{m_0},\dots,a_{m_1}\} \xleftrightarrow{Z} X(z)=\sum_{m_0}^{m_1} a_i z^{-i}$$



ESEMPIO: calcolare $Z\{x(n)=\{5,\underset{\uparrow}{1},-2,3\}\}$

$$X(z)=\sum_{-1}^{2} a_i z^{-i} = 5z+1-2z^{-1}+3z^{-2}$$

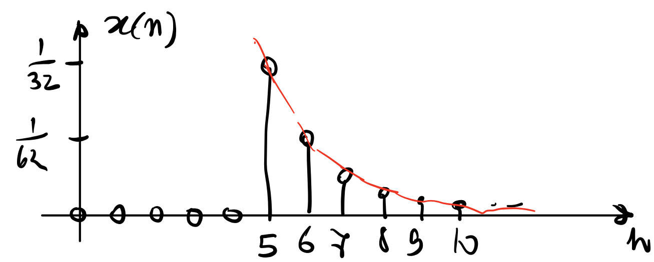
ESERCIZI di calcolo della trasformata Z:



★ $x(n) = n\cdot u(n) = \begin{cases} n & n\geq 0 \\ 0 & n<0 \end{cases}$

$$x(n)=n\cdot u(n) \xleftrightarrow{Z} X(z) = -z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right) = -z\frac{d}{dz}\frac{z}{z-1} = -z\frac{1(z-1)-z(1)}{(z-1)^2} =$$

$$= -z\frac{\cancel{z}-1-\cancel{z}}{(z-1)^2} = \frac{z}{z^2-2z+1} = \frac{z^{-1}}{1-2z^{-1}+z^{-2}}\ ;\quad \text{ROC}: \text{Roc}\left[\frac{1}{1-z^{-1}}\right]: |z|>1$$



★ $x(n)=\begin{cases}\left(\frac{1}{2}\right)^n & \text{per } n\geq 5 \\ 0 & \text{per } n<5\end{cases}$

$$x(n)=\{\underset{\uparrow}{0},0,0,0,0,\tfrac{1}{32},\tfrac{1}{64},\tfrac{1}{128},\dots\}$$

$$x(n)=\begin{cases}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{n-5} & \text{per } n\geq 5 \\ 0 & \text{altrove}\end{cases}$$

Considero: $x'(n)=x(n+5)=\begin{cases}\frac{1}{32}\left(\frac{1}{2}\right)^n & \text{per } n\geq 0 \\ 0 & \text{altrove}\end{cases} = \frac{1}{32}\left(\frac{1}{2}\right)^n u(n)$

$$\rightarrow X'(z)=\frac{1}{32}\,\frac{1}{1-\frac{1}{2}z^{-1}} = \frac{1}{32-16z^{-1}} \rightarrow \boxed{X(z)=X'(z)\cdot z^{-5} = \frac{z^{-5}}{32-16z^{-1}}}$$

$$\boxed{\text{ROC}: |z|>\tfrac{1}{2}\ ;\ z\neq 0}$$

CALCOLO di ANTITRASFORMATA z (casi comuni)

★ SEQUENZE di LUNGHEZZA FINITA : vedi sopra

$$X(z) = \sum_{n_0}^{n_1} a_i z^{-i} \xleftrightarrow{z} x(n) = \{a_{n_0}, a_{n_0+1}, \dots, a_{n_1}\}$$

★ TRASFORMATE in forma RAZIONALE (rapporto di POLINOMI)

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots a_N z^{-N}} = \frac{B(z^{-1})}{A(z^{-1})} \begin{array}{l} \leftarrow \text{GRADO } M \\ \leftarrow \text{GRADO } N \end{array}$$

Abbiamo 2 CASI :

1) $M < N$: FRAZIONE PROPRIA

$$X(z) = \frac{B(z)}{1 + a_1 z^{-1} + \dots a_N z^{-N}} = \frac{B(z)}{\prod_1^N (1 - a_i z^{-1})}$$

N RADICI di $A(z)$

$$x^2 - 2x + 1 = (x-1)(x-1) \quad x=1 \quad x=1$$

$$x^2 - 1 = (x-1)(x+1) \quad x=1 \quad x=-1$$

SCOMPOSIZIONE in FRAZIONI PARZIALI :

$$\frac{B(z)}{\prod_1^N (1 - p_i z^{-1})} = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \dots + \frac{A_N}{1 - p_N z^{-1}} = \sum_1^N \frac{A_i}{1 - p_i z^{-1}}$$

Se : $X(z) = \sum_1^N \frac{A_i}{1 - p_i z^{-1}} \xrightarrow{z^{-1}} x(n) = \sum_1^N A_i \, p_i^n \, u(n)$

2) FRAZIONE IMPROPRIA : $M \geq N$

↳ SOMMA di un POL. INTERO + FRAZIONE PROPRIA

$$X(z) = \frac{B_M(z)}{A_N(z)} = c_0 + c_1 z^{-1} + \dots + c_r z^{-r} + \frac{B_{N-1}(z)}{A_N(z)}, \quad r = M - N : \text{GRADO RELATIVO}$$

PROPRIA

$$= \underbrace{c_0 + c_1 z^{-1} + \dots + c_r z^{-r}}_{\text{seq. LUNGHEZZA FINITA}} + \underbrace{\sum_1^N \frac{A_i}{1 - p_i z^{-1}}}_{\text{FRAZ. PROPRIA}}$$

$c_0 \delta(n) + c_1 \delta(n-1) \dots + c_r \delta(n-r)$

$$\xrightarrow{Z^{-1}} x(n) = c_0\delta(n)+c_1\delta(n-1)+\dots c_r\delta(n-r)+\sum_{i=1}^{N} A_i\, p_i^{\,n}\, u(n)$$

ESEMPIO:

$$X(z)=\frac{z^2}{1-3z+2z^2}=\frac{1}{2-3z^{-1}+z^{-2}} \quad \begin{matrix}\leftarrow M=0\\ \leftarrow N=2\end{matrix} \rightarrow M<N : \text{Frazione PROPRIA.}$$

FATTORIZZO il DENOMINATORE:

$$2-3z^{-1}+z^{-2} = \underbrace{2-2z^{-1}}\underbrace{-z^{-1}+z^{-2}} = 2(1-z^{-1})-z^{-1}(1-z^{-1}) = (1-z^{-1})(2-z^{-1})$$

$$X(z)=\frac{1}{(1-z^{-1})(2-z^{-1})}=\frac{A}{1-z^{-1}}+\frac{B}{2-z^{-1}}=\frac{A(2-z^{-1})+B(1-z^{-1})}{(1-z^{-1})(2-z^{-1})}$$

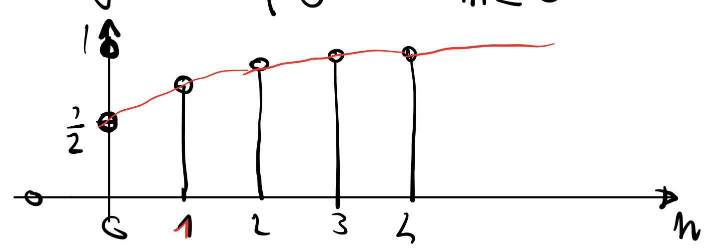
$$\rightarrow 1 = A(2-z^{-1})+B(1-z^{-1}) = 2A-Az^{-1}+B-Bz^{-1} = \underbrace{(2A+B)}_{1}-\underbrace{(A+B)}_{0}z^{-1}$$

$$\rightarrow \begin{cases}2A+B=1 & 2A-A=1 \rightarrow A=1\\ A+B=0 \rightarrow B=-A & \rightarrow B=-1\end{cases}$$

$$\rightarrow X(z)=\frac{1}{1-z^{-1}}-\frac{1}{2-z^{-1}}=\frac{1}{1-z^{-1}}-\frac{1}{2}\,\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$\rightarrow x(n)=u(n)-\frac{1}{2}\left(\frac{1}{2}\right)^n u(n)=\left[1-\left(\frac{1}{2}\right)^{n+1}\right]u(n)=\begin{cases}1-\left(\frac{1}{2}\right)^{n+1} & n\geq 0\\ 0 & n<0\end{cases}$$

$$x(n)=\left\{\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\frac{31}{32},\dots\right\}$$
$\uparrow$



Calcolare l'ANTITRASFORMATA di:

$$X(z)=\frac{1+2z^{-1}-z^{-2}+z^{-3}}{1-z^{-2}} \quad \begin{matrix}\leftarrow M=3\\ \leftarrow N=2\end{matrix}$$

frazione IMPROPRIA → POLIN. INTERO + FRAZ. PROPRIA

$$=\frac{1-z^{2}+2z^{-1}+z^{-3}}{1-z^{-2}}=\frac{1-z^{-2}}{1-z^{-2}}+\frac{2z^{-1}+z^{3}}{1-z^{-2}}=1+\frac{2z^{-1}+z^{-3}}{1-z^{-2}} \quad \begin{matrix}\leftarrow M=2\\ \leftarrow N=2\end{matrix}$$

$$= 1 + \frac{3z^{-1} - z^{-1} + z^{-3}}{1-z^{-2}} = 1 + \frac{3z^{-1} - z^{-1}(1-z^{-2})}{1-z^{-2}} = 1 - z^{-1} + \frac{3z^{-1}}{1-z^{-2}}$$

Scompongo in frazioni parziali:

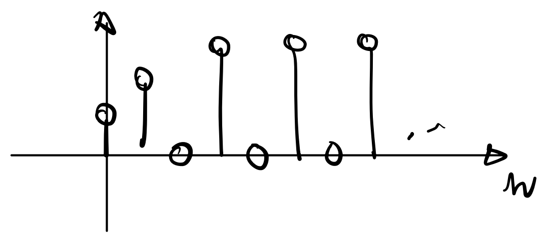
$$\frac{3z^{-1}}{1-z^{-2}} = \frac{3z^{-1}}{(1-z^{-1})(1+z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1+z^{-1}} = \frac{A(1+z^{-1})+B(1-z^{-1})}{(1-z^{-1})(1+z^{-1})}$$

$$\rightarrow A + Az^{-1} + B - Bz^{-1} = 3z^{-1} \Rightarrow \begin{cases} A+B=0 \rightarrow A=-B \\ A-B=3 \rightarrow A+A=2A=3 \end{cases} \quad \boxed{A=\frac{3}{2} \quad B=-\frac{3}{2}}$$

$$\Rightarrow \frac{3z^{-1}}{1-z^{-2}} = \frac{3}{2}\left[\frac{1}{1-z^{-1}} - \frac{1}{1+z^{-1}}\right] \xrightarrow{Z^{-1}} \frac{3}{2}\left[u(n) - (-1)^n u(n)\right]$$

$$\rightarrow x(n) = \delta(n) - \delta(n-1) + \frac{3}{2}\left(1-(-1)^n\right)u(n)$$

$$x(n) = \{\underset{\uparrow}{1}, 2, 0, 3, 0, 3, 0, 3, \dots\}$$



SISTEMI DISCRETI nel DOMINIO Z

$$x(n) \longrightarrow \boxed{\begin{array}{c}\text{SISTEMA LTI} \\ \text{DISCRETO} \\ h(n)\end{array}} \longrightarrow y(n) = x(n) * h(n) = \sum_{i=-\infty}^{\infty} x(i)\,h(n-i)$$

$$x(n) \xrightarrow{Z} X(z) \qquad h(n) \xrightarrow{Z} H(z) \qquad y(n) \xrightarrow{Z} \boxed{Y(z) = X(z)\cdot H(z)}$$

$H(z) = Z\{h(n)\}$: FUNZIONE di TRASFERIMENTO

POLI e ZERI della F. di TRASFERIMENTO

Data $H(z)$, si dicono:

- ☆ ZERI di $H(z)$: i valori di z per cui $H(z)=0$ ZERI: $z_i \in \mathbb{C}$ t.c. $H(z=z_i)=0$
- ☆ POLI di $H(z)$: i valori di z per cui $H(z)=\pm\infty$ POLI: $p_i \in \mathbb{C}$ t.c. $H(z=p_i)=\pm\infty$

Se $H(z)$ è espressa in FORMA RAZIONALE:

$$H(z) = \frac{N_m(z^{-1})}{D_n(z^{-1})} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} \quad \begin{array}{l} \longrightarrow m \text{ RADICI} \\ \longrightarrow n \text{ RADICI} \end{array}$$

$N_m(z^{-1}) = 0 \longrightarrow z = \{z_1, z_2, \dots, z_m\}$ $\quad H(z = z_i) = 0 \to \{z_i\}$: m ZERI di $H(z)$

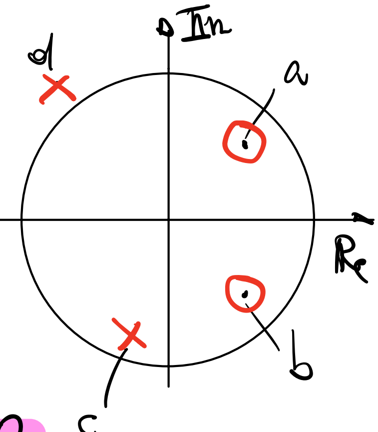
$D_n(z^{-1}) = 0 \longrightarrow z = \{p_1, p_2, \dots, p_n\}$ $\quad H(z = p_i) = \pm\infty \to \{p_i\}$: n POLI di $H(z)$

$$\boxed{H(z) = \frac{N_m(z^{-1})}{D_n(z^{-1})} = b_0 \frac{\prod_1^m (1 - z_i z^{-1})}{\prod_1^n (1 - p_i z^{-1})} \quad \begin{array}{l} \longrightarrow \{z_i\}: \text{ZERI di } H(z) \\ \longrightarrow \{p_i\}: \text{POLI di } H(z) \end{array}}$$

DIAGRAMMA POLI-ZERI:

rappr. grafica di POLI e ZERI di $H(z)$ sul PIANO di GAUSS:

- POLI: SIMBOLO: ×
- ZERI: SIMBOLO: O



Esempio: $X(z) = \dfrac{(1 - a z^{-1})(1 - b z^{-1})}{(1 - c z^{-1})(1 - d z^{-1})}$ $\begin{array}{l} \longrightarrow \text{ZERI}: \{a, b\} \\ \longrightarrow \text{POLI}: \{c, d\} \end{array}$

CAUSALITÀ:

Se un SISTEMA LTI è CAUSALE $\longrightarrow$ $h(n) = 0$ per $n < 0$

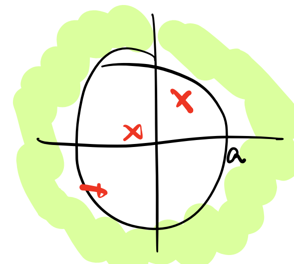
Nel DOMINIO Z: considero $H(z)$

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{-\infty}^{\infty} h(n) z^{-n} = \left\{\begin{matrix} h(n) = 0 \\ n < 0 \end{matrix}\right\} = \sum_{0}^{\infty} h(n) z^{-n}$$

← z ha SOLO ESPONENTI NEGATIVI

→ $H(z)$: SERIE di POTENZE con SOLI ESPONENTI NEGATIVI

→ ROC: $|z| > |a| = f(h(n))$



SISTEMA LTI CAUSALE $\longrightarrow$ ROC$[H(z)]$: $|z| > |p_{MAX}|$

dove $p_{MAX} = \max\{|p_i|\}$

STABILITÀ (BIBO)

SISTEMA LTI è STABILE sse $\sum_{-\infty}^{\infty} |h(n)| < \infty$ (val. FINITO)

Nel dominio z ?

Considero: $|H(z)|_{|z|=1} = \left| \sum_{-\infty}^{\infty} h(n) z^{-n} \right| \leq \sum_{-\infty}^{\infty} \left| h(n) z^{-n} \right| = \{|z|=1\} =$

$= \sum_{-\infty}^{\infty} |h(n)| \, |\underbrace{z^{-n}}_{1}| = \sum_{-\infty}^{\infty} |h(n)| < \infty$ FINITO $\rightarrow$ H(z) CONVERGE !

$\rightarrow$ Se SISTEMA LTI è STABILE $\rightarrow$ H(z) CONVERGE in $|z|=1$

Se S[·] è CAUSALE $\rightarrow$ ROC: $|z| > p_{max}$
Se S[·] è STABILE $\rightarrow$ $|z| = 1 \in$ ROC
$\Rightarrow p_{max} = \max\{p_i\} < 1$

Un SISTEMA LTI è CAUSALE e STABILE (BIBO) SSE TUTTI i POLI p_i di H(z) sono CONTENUTI nel CERCHIO UNITARIO

RISPOSTA in FREQUENZA

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$

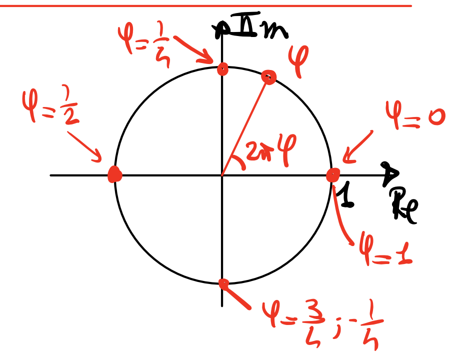
RISPOSTA in FREQUENZA $= \text{DTFT}\{h(n)\} = H_{DTFT}(\varphi) = \sum_{-\infty}^{\infty} h(n) e^{-j2\pi n\varphi}$, $0 \leq \varphi < 1$

FUNZ. di TRASFERIMENTO: $H(z) = \mathcal{Z}\{h(n)\} = \sum_{-\infty}^{\infty} h(n) z^{-n}$

$$\boxed{\text{RISPOSTA in FREQUENZA: } H_{DTFT}(\varphi) = H(z),\ z = e^{j2\pi\varphi}, \quad 0 \leq \varphi < 1}$$

Dato H(z), se voglio $H_{DTFT}(\varphi)$

$\rightarrow H(z = e^{j2\pi\varphi}) = H(\varphi)$, $0 \leq \varphi < 1$



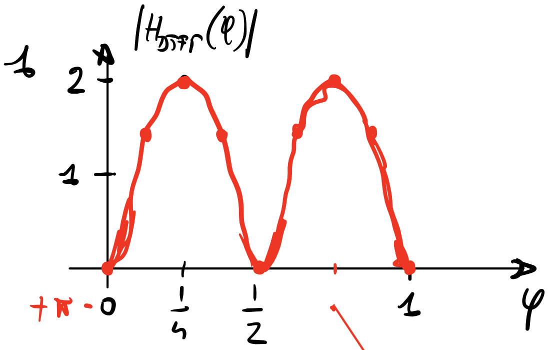
ESEMPIO:
Calcolare la risposta in frequenza di $H(z) = 1 - z^{-2}$

$$H_{DTFT}(\varphi) = H(z = e^{j2\pi\varphi}) = 1 - e^{-j4\pi\varphi} \quad \leftarrow \text{RISPOSTA in FREQUENZA}$$

RISPOSTA in AMPIEZZA $= \left|H_{DTFT}(\varphi)\right| = \left|1 - e^{-j4\pi\varphi}\right| = \left|\underbrace{1 - \cos(4\pi\varphi)}_{Re} - j\underbrace{\sin(-4\pi\varphi)}_{Im}\right|$

$$= \sqrt{(1-\cos(4\pi\varphi))^2 + \sin^2(4\pi\varphi)} = \sqrt{1 - 2\cos(4\pi\varphi) + \underbrace{\cos^2(4\pi\varphi) + \sin^2(4\pi\varphi)}_{1}} =$$

$$= \sqrt{2 - 2\cos(4\pi\varphi)} = \sqrt{2}\sqrt{1-\cos(4\pi\varphi)}, \quad 0 \le \varphi < 1$$

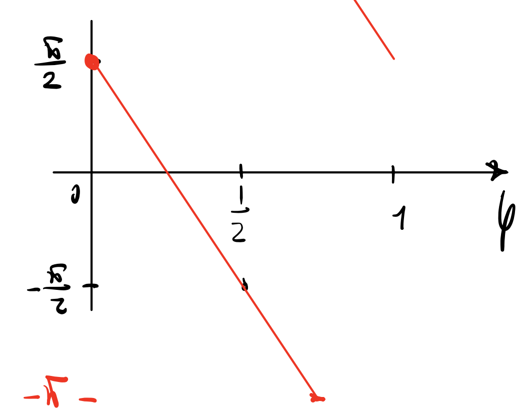


RISPOSTA di FASE:

$$\measuredangle H(\varphi) = \operatorname{atan} \frac{Im[H(\varphi)]}{Re[H(\varphi)]} = \operatorname{atan}\left[\frac{\sin(4\pi\varphi)}{1-\cos(4\pi\varphi)}\right] =$$

$$= \operatorname{atan}(\operatorname{cotg}(2\pi\varphi)) = \operatorname{atan}\left(\tan\left(\frac{\pi}{2} - 2\pi\varphi\right)\right)$$

$$= \frac{\pi}{2} - 2\pi\varphi$$



ANALISI di SISTEMI LTI DISCRETI (FILTRI NUMERICI)

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n) = \sum_{i=-\infty}^{\infty} h(i)\,x(n-i)$$

$$X(z) \qquad H(z) \qquad Y(z) = X(z)\cdot H(z)$$

Considero SISTEMI CAUSALI: $h(n) = 0$ per $n < 0$

$$y(n) = \sum_{i=0}^{\infty} h(i)\,x(n-i)$$

SISTEMI con RISPOSTA ALL'IMPULSO di DURATA FINITA

(**FILTRI FIR: FINITE IMPULSE RESPONSE**)

$h(n)$ è CAUSALE e di DURATA D $\rightarrow$ $h(n) = 0$ per: $\begin{cases} n<0 \\ n>D \end{cases}$

$$y(n) = x(n) * h(n) = \sum_{0}^{D} {}_i\, h(i)\, x(n-i) = \underbrace{h(0)x(n)}_{\text{INGRESSO ATTUALE}} + \underbrace{h(1)x(n-1) + \dots + h(D)x(n-D)}_{D \text{ INGRESSI PASSATI}}$$

Nel DOMINIO Z:

$$h(n) = \{\underset{\uparrow}{h(0)}, h(1), \dots, h(D)\} \xrightarrow{Z} \boxed{H(z) = \sum_{0}^{D} {}_i\, h(i)\, z^{-i}} \quad \text{FUNZIONE di TRASFERIMENTO}$$

Dato $x(n) \xrightarrow{Z} X(z) \longrightarrow Y(z) = X(z) \cdot H(z) \longrightarrow y(n) = Z^{-1}\{Y(z)\}$

ESEMPIO: calcolare l'USCITA del FILTRO con $h(n) = \{\underset{\uparrow}{1}, -1\}$ $h(0)$ $h(1)$

per il segnale d'INGRESSO: $x(n) = \{\underset{\uparrow}{1}, 2, 1\}$ $x(0)$ $x(1)$ $x(2)$

Nei TEMPI:

$$y(n) = \sum_{0}^{1} {}_i\, h(i)\, x(n-i) = h(0)x(n) + h(1)x(n-1) = x(n) - x(n-1)$$

Uscite: per

$n = 0 \longrightarrow y(0) = x(0) - x(-1) = 1 - 0 = 1$

$n = 1 \longrightarrow y(1) = x(1) - x(0) = 2 - 1 = 1$

$n = 2 \longrightarrow y(2) = x(2) - x(1) = 1 - 2 = -1$

$n = 3 \longrightarrow y(3) = x(3) - x(2) = 0 - 1 = -1$

$n \geq 4 \longrightarrow y(n \geq 4) = x(\geq 4) - x(\geq 3) = 0$

$$\longrightarrow y(n) = \{\underset{\uparrow}{1}, 1, -1, -1\}$$

ANALISI nel DOMINIO Z:

$$x(n) = \{\underset{\uparrow}{1}, 2, 1\} \xrightarrow{Z} X(z) = 1z^{0} + 2z^{-1} + 1z^{-2} = 1 + 2z^{-1} + z^{-2}$$

$$h(n) = \{\underset{\uparrow}{1}, -1\} \xrightarrow{Z} H(z) = 1 - z^{-1}$$

$$Y(z) = X(z) \cdot H(z) = (1 + 2z^{-1} + z^{-2})(1 - z^{-1}) = 1 + 2z^{-1} + z^{-2} - z^{-1} - 2z^{-2} - z^{-3} =$$

$$= 1 + z^{-1} - z^{-2} - z^{-3}$$

$$\longrightarrow y(n) = \mathcal{Z}^{-1}\{1+z^{-1}-z^{-2}-z^{-3}\} = \delta(n)+\delta(n-1)-\delta(n-2)-\delta(n-3) =$$

$$= \{\underset{\uparrow}{1}, 1, -1, -1\}$$

SISTEMI LTI con RISPOSTA ALL'IMPULSO di DURATA INFINITA

FILTRI IIR - INFINITE IMPULSE RESPONSE

Considero $h(n) = u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

$$\longrightarrow y(n) = x(n) * h(n) = \sum_{i=0}^{\infty} h(i)\,x(n-i) = \sum_{i=0}^{\infty} x(n-i)$$

in FORMA CANONICA

Posso scrivere:

$$y(n) = \sum_{i=0}^{\infty} x(n-i) = x(n) + \sum_{i=1}^{\infty} x(n-i) = x(n) + \sum_{i=0}^{\infty} x(n-(i+1)) =$$

$$= x(n) + \sum_{i=0}^{\infty} x(n-1-i) = x(n) + y(n-1)$$

Se conosco un VALORE INIZIALE per $y(n)$ $\left(y(n=0) = y_0 \right)$

$$\longrightarrow \boxed{y(n) = \begin{cases} y_0 & \text{per } n = 0 \\ x(n) + y(n-1) & \text{per } n > 0 \end{cases}}$$

Eq. in FORMA RECURSIVA

Nel dominio z: la trasformata Z di una sequenza INFINITA è, in genere, l'espressione, nel dominio trasformato, della FORMA RECURSIVA

$$\longrightarrow h(n) = u(n) \xrightarrow{\mathcal{Z}} X(z) = \frac{1}{1-z^{-1}}$$

$$Y(z) - Y(z)z^{-1} = X(z)$$

In Z: $Y(z) = H(z) \cdot X(z) = \dfrac{X(z)}{1-z^{-1}} \longrightarrow \boxed{Y(z)\left(1-z^{-1}\right) = X(z)}$

Antitrasformo: $y(n) - y(n-1) = x(n)$

$$\rightarrow \boxed{y(n) = x(n) + y(n-1)}$$ Forma RECURSIVA

FILTRO IIR : CASO GENERALE

espresso con $H(z)$ in FORMA RAZIONALE:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 - a_1 z^{-1} \dots - a_N z^{N}} = \frac{\sum_0^M{}_i \, b_i z^{-i}}{1 - \sum_1^N{}_i \, a_i z^{-i}}$$

← parte FIR
← parte IIR

Antitrasformo $H(z)$, separando $X(z)$ e $Y(z)$:

$$Y(z)\left[1 - a_1 z^{-1} - \dots - a_N z^{-N}\right] = X(z) \cdot \sum_0^M{}_i \, b_i z^{-i}$$

Isolo $Y(z)$: $$Y(z) = Y(z) \sum_1^N{}_i \, a_i z^{-i} + X(z) \sum_0^M{}_i \, b_i z^{-i}$$

Antitrasformo:

$$\underset{\substack{\downarrow \\ \text{USCITA ATTUALE}}}{y(n)} = \underbrace{\sum_1^N{}_i \, a_i \, y(n-i)}_{\text{N USCITE PRECEDENTI}} + \underbrace{\sum_0^M{}_i \, b_i \, x(n-i)}_{\text{M INGRESSI PASSATI + INGRESSO ATTUALE}}$$

Eq. RECURSIVA di un GENERICO FILTRO IIR

Un GENERICO filtro IIR si può DESCRIVERE con DUE FORME dell'EQUAZIONE ALLE DIFFERENZE

Forma CANONICA: $$y(n) = \sum_0^\infty{}_i \, h(i) x(n-i)$$

Forma RECURSIVA: $$y(n) = \sum_1^N{}_i \, a_i \, y(n-i) + \sum_0^M{}_i \, b_i \, x(n-i)$$

STABILITÀ dei FILTRI IIR / FIR

$\boxed{\text{FIR}}$: $h(n) = \{h(0) \dots h(M)\} \rightarrow y(n) = \sum_0^M{}_i \, h(i) x(n-i)$

$$\rightarrow H(z) = \sum_0^M{}_i \, h(i) z^{-i} = b_0 + b_1 z^{-1} \dots + b_M z^{-M} = b_0 \prod_1^M{}_i \left(1 - z_i z^{-1}\right)$$

z_i, $i = 1 \dots M$: M ZERI di $H(z)$; NO POLI

→ FIR è SEMPRE STABILE

IIR: $H(z) = \dfrac{\sum_{i=0}^{M} b_i z^{-i}}{1 - \sum_{i=1}^{N} a_i z^{-i}} = \dfrac{b_0 \prod_i (1 - z_i z^{-1})}{\prod_{i=1}^{N} (1 - p_i z^{-1})}$

→ M ZERI del FILTRO

→ N POLI

→ IIR è STABILE sse $|p_i| < 1, \forall p_i \quad i = 1, .. N$
(TUTTI i POLI DENTRO le CIRCONF. UNITARIA).

SINTESI (o PROGETTO) di FILTRI DIGITALI

In forma "standard": IMPLEMENTAZIONE dell'EQUAZ. alle DIFFERENZE

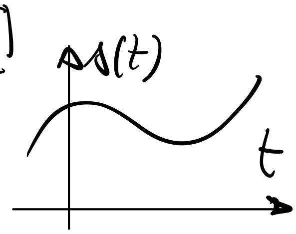
FIR: $y(n) = \sum_{i=0}^{M} b_i \, x(n-i)$

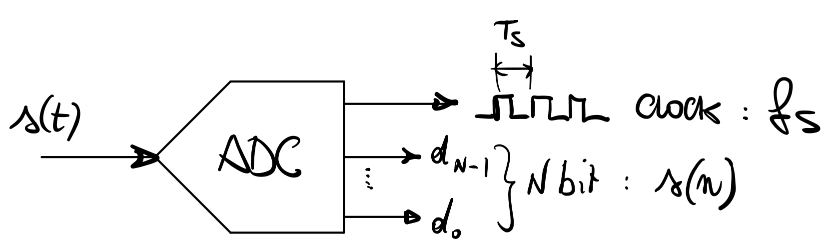
IIR: $y(n) = \sum_{i=1}^{N} a_i \, y(n-i) + \sum_{i=0}^{M} b_i \, x(n-i)$ (forma RECURSIVA)

CHE COSA CI SERVE?

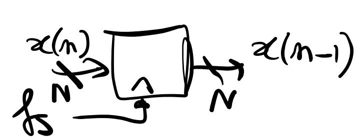
- SEGNALI: $x(n)$
  → SEGNALI PASSATI: $x(n-1), \ldots, x(n-M)$; $y(n-1), \ldots, y(n-N)$
- SOMMATORI
- MOLTIPLICATORI

SEGNALI



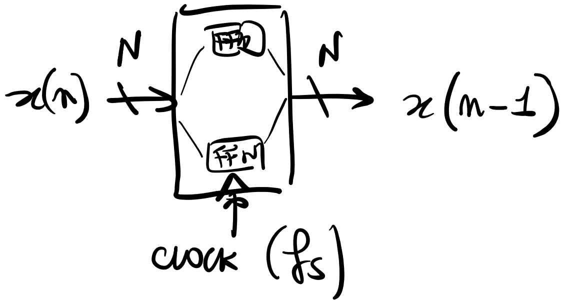


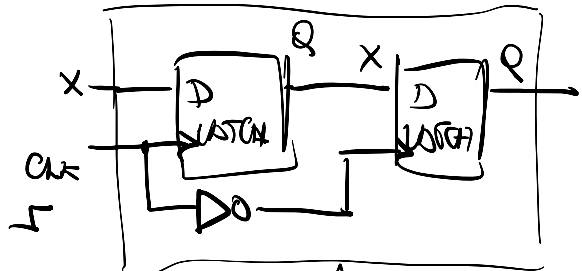
CAMPIONI PASSATI → "RITARDATI"



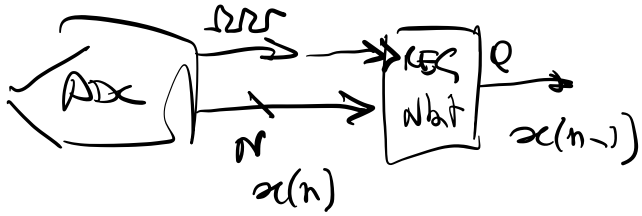
RITARDATORE

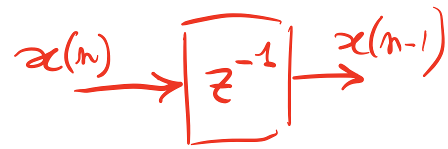
REGISTRO a N bit (N flip-flop tipo D)



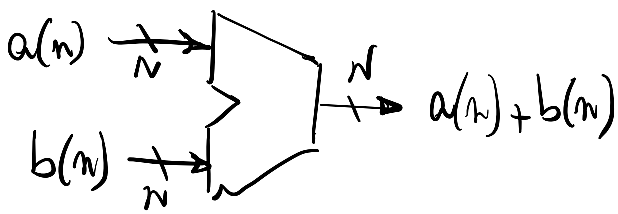


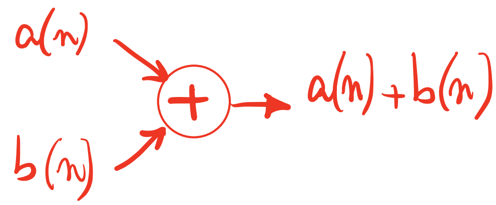
FLIP-FLOP di TIPO D



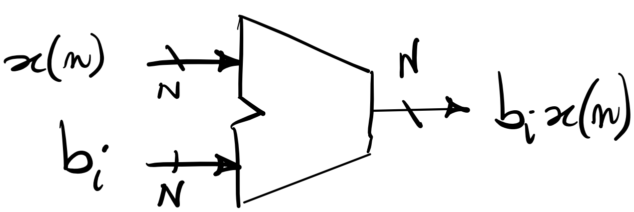


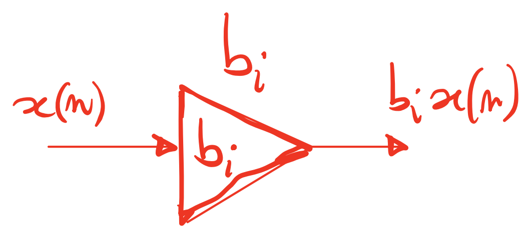
SOMMATORE





MOLTIPLICATORE



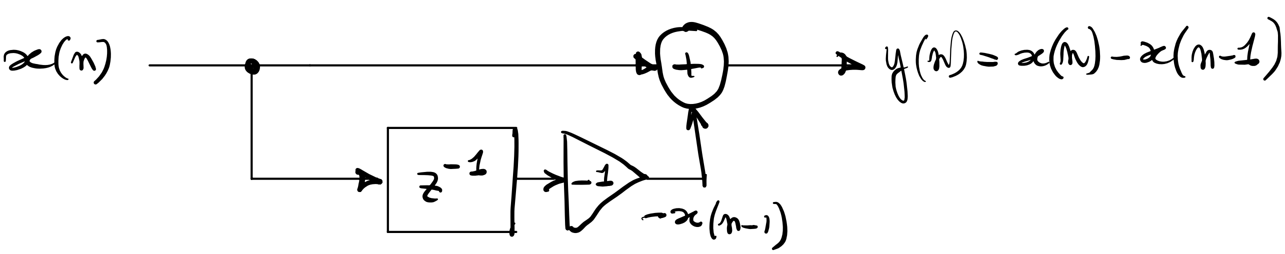


FILTRI FIR : STRUTTURA CIRCUITALE

Esempio : consideriamo : $h(n) = \{\underset{\uparrow}{1}, -1\} \rightarrow H(z) = 1 - z^{-1}$

$$y(n) = \sum_{i=0}^{1} h(i)\,x(n-i) = x(n) - x(n-1)$$

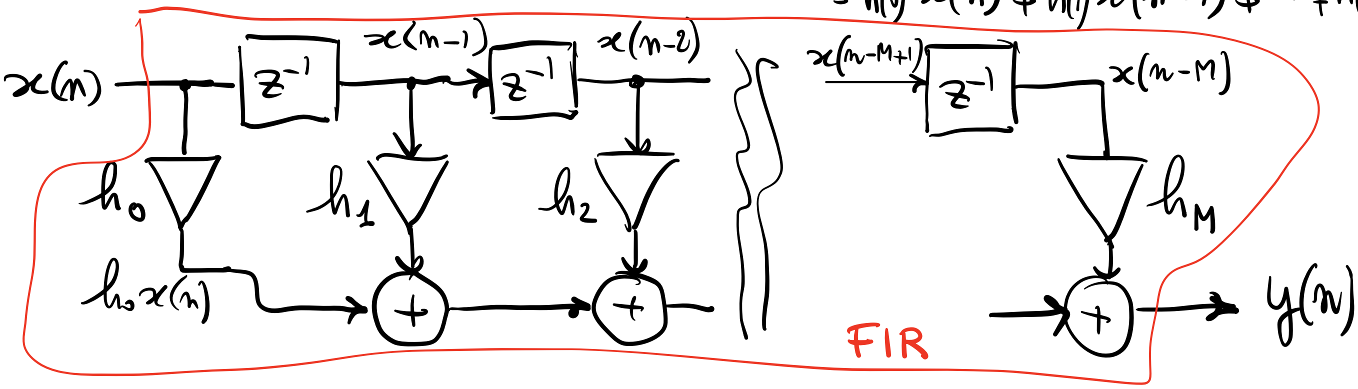
$$Y(z) = \qquad = X(z) - z^{-1}X(z)$$



Caso GENERALE : FIR GENERICO :

$$h(n) = \{h(0), \dots, h(M)\} \longrightarrow y(n) = \sum_{i=0}^{M} h(i)\, x(n-i) =$$

$$= h(0)x(n) + h(1)x(n-1) + \dots + h(M)x(n-M)$$



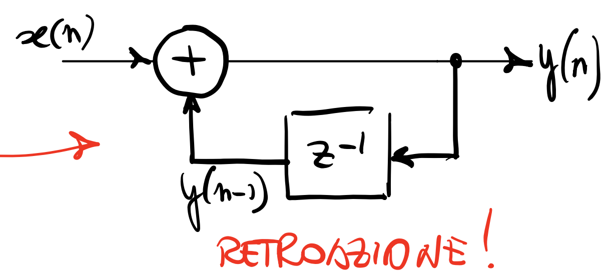
FILTRI IIR : STRUTTURA CIRCUITALE

$$y(n) = \sum_{i=1}^{N} a_i\, y(n-i) + \underbrace{\sum_{i=0}^{M} b_i\, x(n-i)}_{v(n)}$$

$$= a_1 y(n-1) + a_2 y(n-2) + \dots + a_N y(n-N) + v(n) \quad ; \quad v(n) = \sum_{i=0}^{M} b_i\, x(n-i)$$

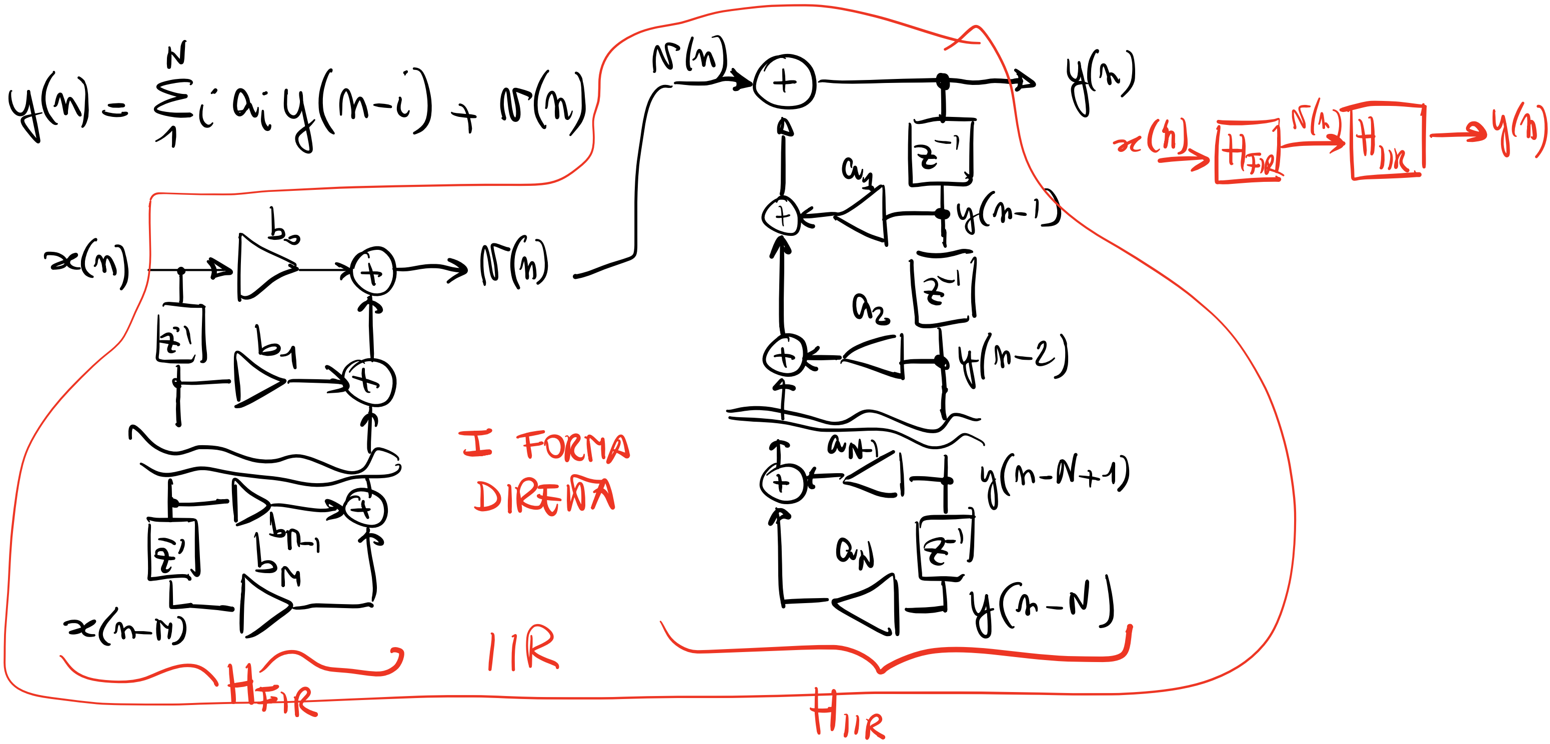
Esempio: $y(n) = y(n-1) + x(n)$

forma RECURSIVA

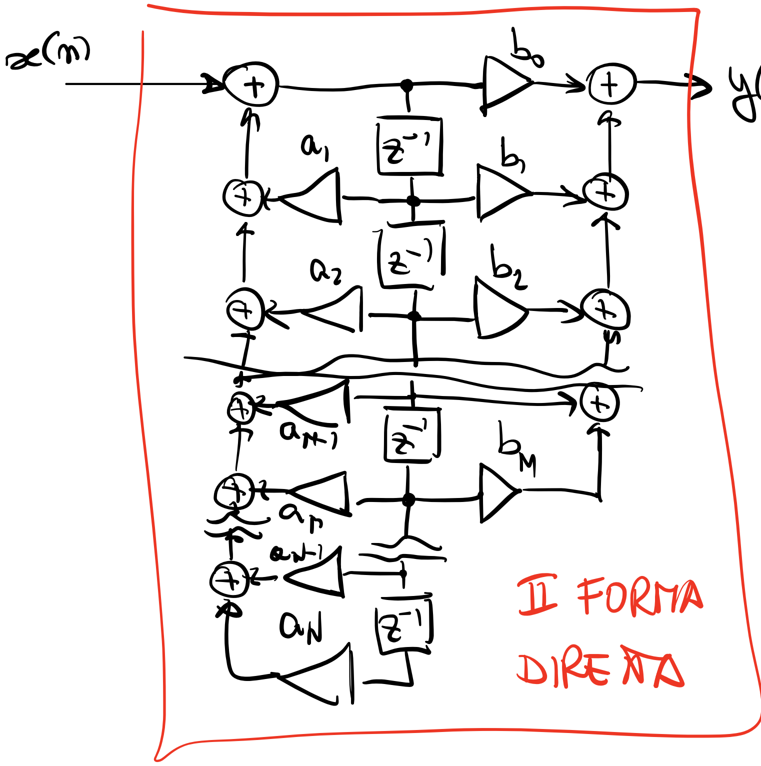


Caso GENERALE - IIR GENERICO

$$y(n) = \sum_{i=1}^{N} a_i\, y(n-i) + v(n)$$



Invertendo le posizioni di H_{FIR} e H_{IIR} ottengo:



$$M < N$$

$\max(N, M)$ RITARDATORI

$M + N$ SOMMATORI

$N + M + 1$ MOLTIPLICATORI

ESEMPI di FILTRI DIGITALI

FILTRO a PETTINE (COMB FILTER)

Famiglia di FILTRI FIR

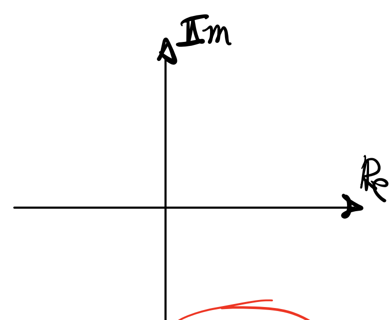
FILTRO a PETTINE di ORDINE M: $\boxed{H(z) = 1 - z^{-M}}$

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-M} \longrightarrow Y(z) = (1 - z^{-M})\, X(z) = X(z) - z^{-M} X(z)$$

z^{-1}

$$\boxed{y(n) = x(n) - x(n-M)}$$

DIAGRAMMA POLI-ZERI:



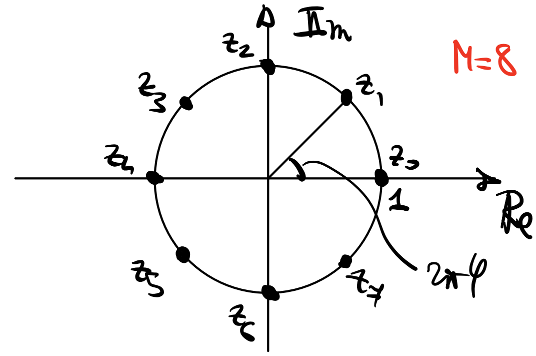
- POLI: NO POLI
- ZERI: M ZERI: le RADICI di: $1 - z^{-M} = 0$

Risolviamo $1 - z^{-M} = 0 \longrightarrow z^{M} - 1 = 0 \longrightarrow z^{M} = 1 \longrightarrow z = \sqrt[M]{1}$

$z = \sqrt[M]{1} \longrightarrow$ M SOLUZIONI: $\boxed{z_i = \sqrt[M]{1} = e^{j\frac{2\pi}{M} i}, \quad i = 0, \dots, M-1}$

$$\begin{cases} |z_i| = 1 \\ \measuredangle z_i = \frac{2\pi}{M} i \rightarrow \frac{\pi}{4} i \end{cases}$$

M ZERI



ZERI: M VALORI EQUIDISTANTI sulla CIRCONF. di RAGGIO = 1

RISPOSTA in FREQUENZA:

$$\boxed{H(\varphi) = 0 \quad \text{per}: \varphi = \frac{i}{M}, \quad i = 0, \dots, M-1}$$

In generale:

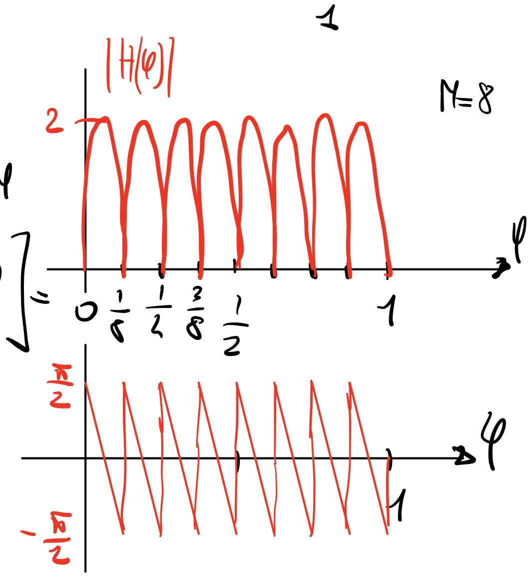
$0 \le \varphi < 1$

$$H_{DTFT}(\varphi) = H(z = e^{j2\pi\varphi}) = 1 - e^{-j2\pi M\varphi} = 1 - \cos(2\pi M\varphi) + j\sin(2\pi M\varphi)$$

RISPOSTA in AMPIEZZA = MODULO di $H(\varphi)$:

$$|H(\varphi)| = \sqrt{(1-\cos(2\pi M\varphi))^2 + \sin^2(2\pi M\varphi)} = \sqrt{1 - 2\cos(2\pi M\varphi) + \underbrace{\cos^2(2\pi M\varphi) + \sin^2(2\pi M\varphi)}_{1}} =$$

$$= \sqrt{2 - 2\cos(2\pi M\varphi)} = \sqrt{2}\sqrt{1 - \cos(2\pi M\varphi)}$$



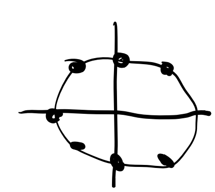
RISPOSTA di FASE: $\measuredangle H(\varphi)$

$\frac{\pi}{2} - \mathrm{tg}\,\pi M\varphi$

$$\measuredangle H(\varphi) = \mathrm{atan2}\left[\frac{\sin(2\pi M\varphi)}{1 - \cos(2\pi M\varphi)}\right] = \mathrm{atan2}\left[\mathrm{cotg}\,\pi M\varphi\right] = \frac{\pi}{2} - \pi M\varphi$$

VARIANTE: CANCELLO lo ZERO in $z=1$ → con un POLO in $z=1$

$$H(z) = \frac{1}{M} \cdot \frac{1 - z^{-M}}{1 - z^{-1}} = \prod_{i=1}^{M-1} (1 - z_i z^{-1}), \quad z_i = e^{j2\pi\frac{i}{M}}$$



$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-M}}{1 - z^{-1}} \longrightarrow Y(z)\left(1 - z^{-1}\right) = X(z)\left(1 - z^{-M}\right)$$

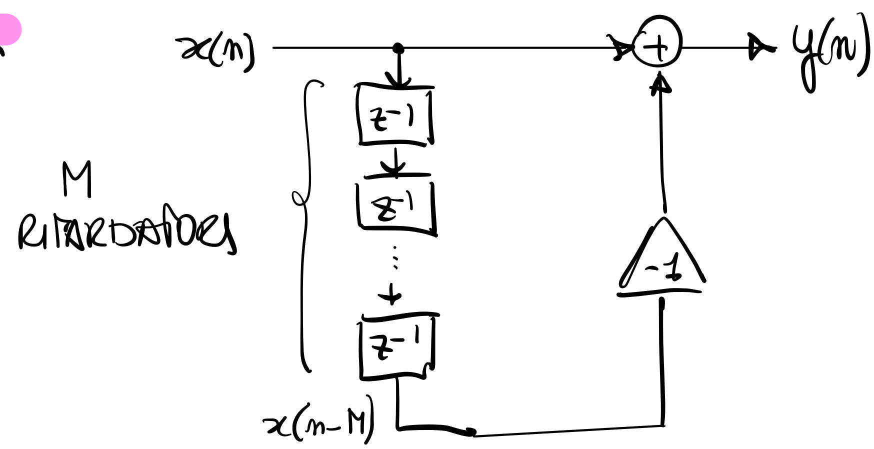
Antitrasformo: $y(n) - y(n-1) = \frac{1}{M}\left[x(n) - x(n-M)\right]$

$$\longrightarrow \boxed{y(n) = y(n-1) + \frac{1}{M}x(n) - \frac{1}{M}x(n-M)}$$

STRUTTURA CIRCUITALE - FILTRI A PETTINE

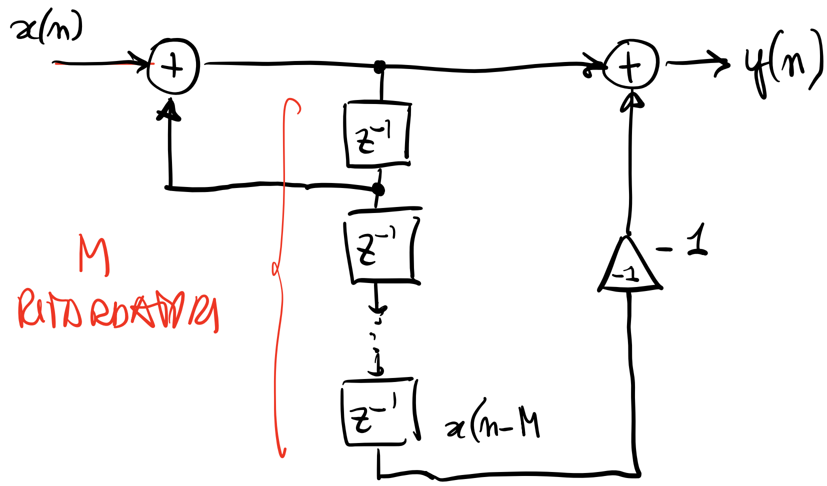
M-ORDER COMB FILTER

$$y(n) = x(n) - x(n-M)$$



M-ORDER, MODIFICATO

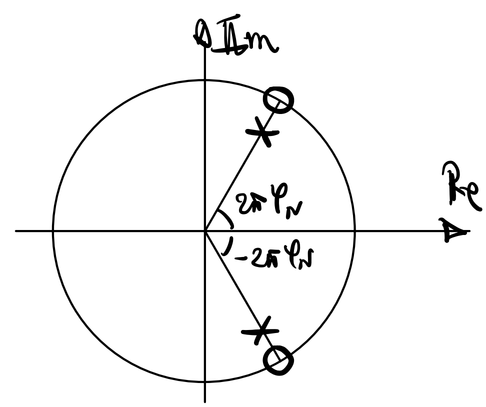
$$y(n) = \underbrace{y(n-1)} + x(n) - x(n-M)$$



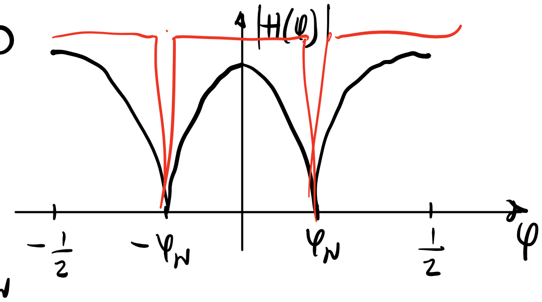
FILTRO "NOTCH" (NOTCH FILTERS)

Famiglia di filtri ARRESTA-BANDA

Date φ_N (freq. da ARRESTARE: $\frac{f_N}{f_S} = \varphi_N$)



→ 2 ZERI a $\pm \varphi_N$ $\qquad H(z = e^{\pm j 2\pi \varphi_N}) = 0$



$z_N = e^{j2\pi\varphi_N}$; $\bar{z}_N = e^{-j2\pi\varphi_N}$

→ 2 POLI, a $\pm\varphi_N$, scelti di α RISPETTO a z_N

$0 \leq \alpha < 1$

$p_N = \alpha z_N = \alpha e^{j2\pi\varphi_N}$; $\bar{p}_N = \alpha \bar{z}_N$, $0 \leq \alpha < 1$

$$\boxed{H(z) = \frac{(1 - z_N z^{-1})(1 - \bar{z}_N z^{-1})}{(1 - p_N z^{-1})(1 - \bar{p}_N z^{-1})} \; ; \quad z_N = e^{j2\pi\varphi_N}, \quad p_N = \alpha z_N = \alpha e^{j2\pi\varphi_N}, \quad 0 \leq \alpha < 1}$$

$$= \frac{1 - (z_N + \bar{z}_N) z^{-1} + z_N \bar{z}_N z^{-2}}{1 - (p_N + \bar{p}_N) z^{-1} + p_N \bar{p}_N z^{-2}} = \frac{1 - 2\,\mathrm{Re}[z_N] z^{-1} + z^{-2}}{1 - 2\alpha\,\mathrm{Re}[z_N] z^{-1} + \alpha^2 z^{-2}} =$$

$$\boxed{H(z) = \frac{1 - 2\cos(2\pi\varphi_N) z^{-1} + z^{-2}}{1 - 2\alpha\cos(2\pi\varphi_N) z^{-1} + \alpha^2 z^{-2}}} = \frac{Y(z)}{X(z)}$$

TUTTI i COEFFICIENTI sono REALI

EQUAZIONE alle DIFFERENZE:

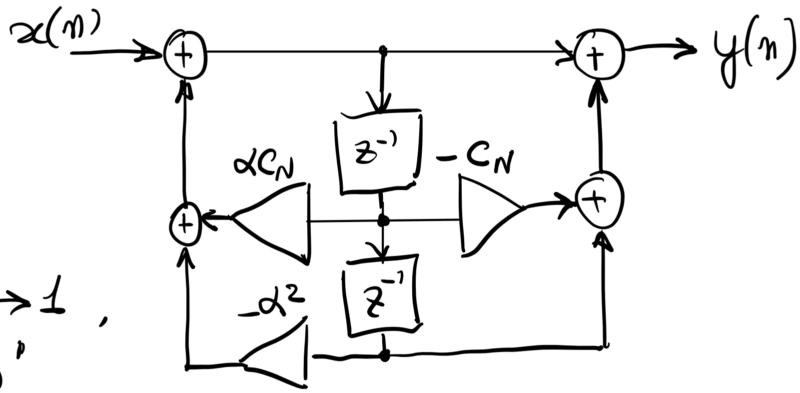
$$Y(z)\left(1 - 2\alpha\cos(2\pi\varphi_N) z^{-1} + \alpha^2 z^{-2}\right) = X(z)\left(1 - 2\cos(2\pi\varphi_N) z^{-1} + z^{-2}\right) \; ; \quad c_N = 2\cos(2\pi\varphi_N)$$

$$\Big\downarrow \mathcal{Z}^{-1}$$

$$y(n) - \alpha c_N y(n-1) + \alpha^2 y(n-2) = x(n) - c_N x(n-1) + x(n-2)$$

$$\boxed{y(n) = \underbrace{\alpha c_N y(n-1) - \alpha^2 y(n-2)}_{\text{IIR}} + \underbrace{x(n) - c_N x(n-1) + x(n-2)}_{\text{FIR}}}$$

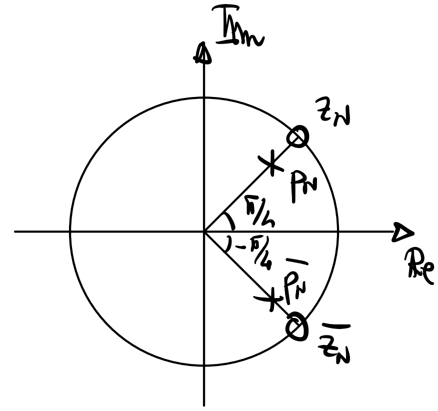
STRUTTURA CIRCUITALE del FILTRO:



α: REGOLA la SELETTIVITÀ del FILTRO NOTCH: quanto più $\alpha \to 1$, tanto più il "notch" è "AFFILATO"

ESERCIZI di RIEPILOGO

PROGETTO filtro NOTCH (Es 4 Tema feb 2023)



$f_N = 50\ Hz$; $f_S = 400\ Hz$; $\alpha = 0,8$

a) $\varphi_N = \dfrac{f_N}{f_S} = \dfrac{50\ Hz}{400\ Hz} = \dfrac{1}{8}$

$z_N = e^{j2\pi\varphi_N} = e^{j2\pi\frac{1}{8}} = e^{j\frac{\pi}{4}}$; $p_N = \alpha z_N = 0,8e^{j\pi/4}$

$$H(z) = \frac{(1-e^{j\frac{\pi}{4}}z^{-1})(1-e^{-j\frac{\pi}{4}}z^{-1})}{(1-0,8e^{j\pi/4}z^{-1})(1-0,8e^{-j\pi/4}z^{-1})} = \frac{1-2\cos\left(\frac{\pi}{4}\right)z^{-1}+z^{-2}}{1-1,6\cos\left(\frac{\pi}{4}\right)z^{-1}+0,64z^{-2}} =$$

$$\boxed{H(z) = \frac{1-\sqrt{2}z^{-1}+z^{-2}}{1-0,8\sqrt{2}z^{-1}+0,64z^{-2}}} = \frac{Y(z)}{X(z)}$$

b) $X(z)\left(1-\sqrt{2}z^{-1}+z^{-2}\right) = Y(z)\left(1-0,8\sqrt{2}z^{-1}+0,64z^{-2}\right)$

$\downarrow \mathcal{Z}^{-}$

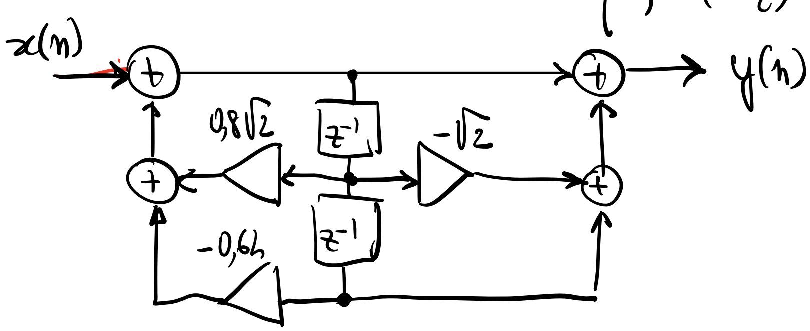
$x(n)-\sqrt{2}x(n-1)+x(n-2) = y(n)-0,8\sqrt{2}y(n-1)+0,64y(n-2)$

Isolo: $\boxed{y(n) = 0,8\sqrt{2}y(n-1)-0,64y(n-2)+x(n)-\sqrt{2}x(n-1)+y(n-2)}$

c) $H\left(\varphi=\frac{200}{400}=\frac{1}{2}\right) = H\left(z=e^{j2\pi\frac{1}{2}}=e^{j\pi}=-1\right) = \dfrac{1+\sqrt{2}+1}{1+0,8\sqrt{2}+0,64} =$

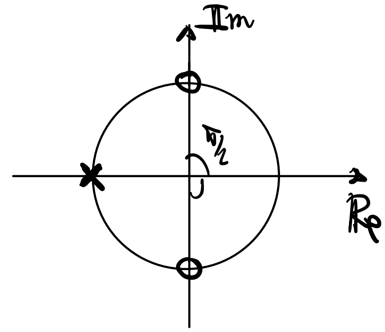
$= \dfrac{2+\sqrt{2}}{1,64+0,8\sqrt{2}} \cong 1,23 \rightarrow \begin{cases} \left|H\left(\varphi=\frac{1}{2}\right)\right| = 1,23 \\ \angle H\left(\varphi=\frac{1}{2}\right) = 0 \end{cases}$

d) CIRCUITO:



FILTRO NUMERICO; $f_s = 300\ Hz$ $f_z = 75\ Hz$ polo: $z = -1$

$$\varphi_z = \frac{f_z}{f_s} = \frac{75}{300} = \frac{1}{4} \longrightarrow z_z = e^{\pm j 2\pi \varphi_z} = e^{\pm j\frac{\pi}{2}} = \pm j$$



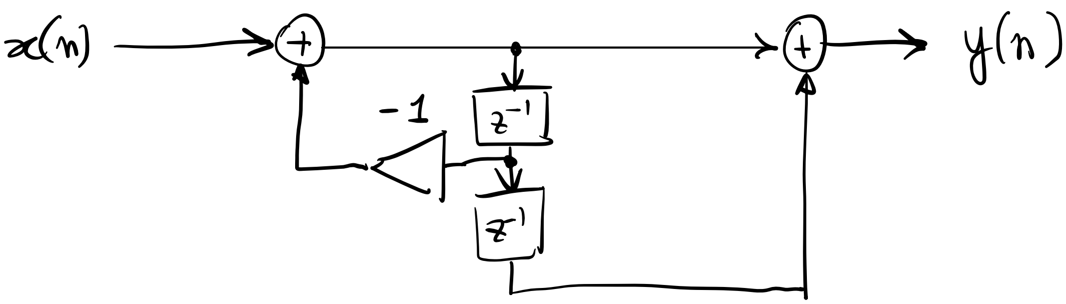
$$H(z) = \frac{(1 - jz^{-1})(1 + jz^{-1})}{(1 + z^{-1})} = \frac{1 + z^{-2}}{1 + z^{-1}} = \frac{Y(z)}{X(z)}$$

$$Y(z)(1 + z^{-1}) = X(z)(1 + z^{-2})$$

$$\downarrow$$

$$y(n) = -y(n-1) + x(n) + x(n-2)$$

$$H_{DTFT}(\varphi = 0) = H(z = e^{j0} = 1) = \frac{1+1}{1+1} = 1 \quad \begin{cases} |H(\varphi=0)| = 1 \\ \measuredangle H(\varphi=0) = 0 \end{cases}$$



$$H(z) = \frac{1 + z^{-2}}{1 + z^{-1}} = \frac{1}{1 + z^{-1}} + \frac{z^{-2}}{1 + z^{-1}}$$

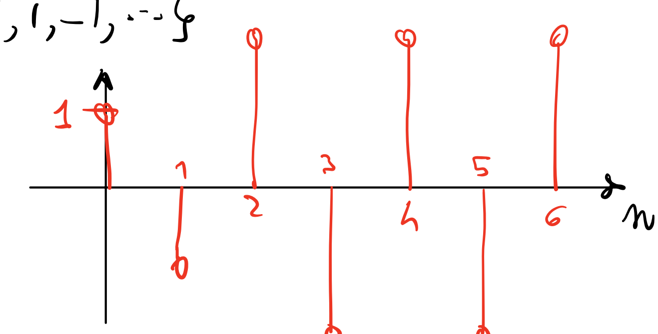
$$\frac{1}{1 - az^{-1}} \longleftrightarrow a^n u(n)$$

$$\downarrow z^{-1}$$

$$h(n) = (-1)^n u(n) + (-1)^{n-2} u(n-2) = \delta(n) - \delta(n-1) + 2(-1)^n u(n-2)$$

$$\{\underset{\uparrow}{1}, -1, 1, -1, \dots\} \qquad \{\underset{\uparrow}{0}, 0, 1, -1, 1, -1, \dots\}$$

$$h(n) = \{1, -1, 2, -2, 2, -2, \dots\}$$



Sviluppo in SERIE di FOURIER:

$s(t)$; $T=4$ $\qquad c_0=2$; $c_2=-j$; $c_{-2}=j$; $c_n=0$, $n\neq 0,\pm 2$

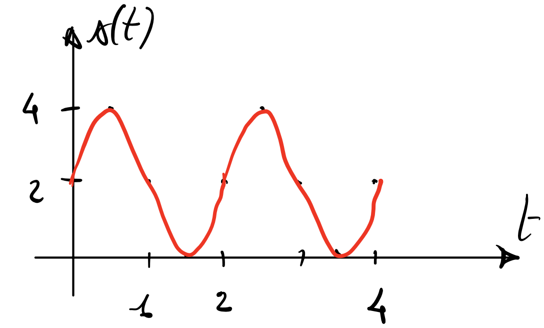
$$s(t)=\sum_{-\infty}^{\infty} c_n e^{j2\pi\frac{n}{T}t} = 2e^0 - je^{j2\pi\frac{1}{2}t} + je^{-j2\pi\frac{1}{2}t} = 2-je^{j\pi t}+je^{-j\pi t}$$

$T \to 4$

$$= 2 + j\left(e^{-j\pi t}-e^{j\pi t}\right) = 2+\underbrace{\frac{1}{j}\left(e^{j\pi t}-e^{-j\pi t}\right)}_{2\sin(\pi t)} =$$

$\sin\theta = \frac{1}{2j}\left(e^{j\theta}-e^{-j\theta}\right)$

$$= 2\left[1+\sin(\pi t)\right]$$

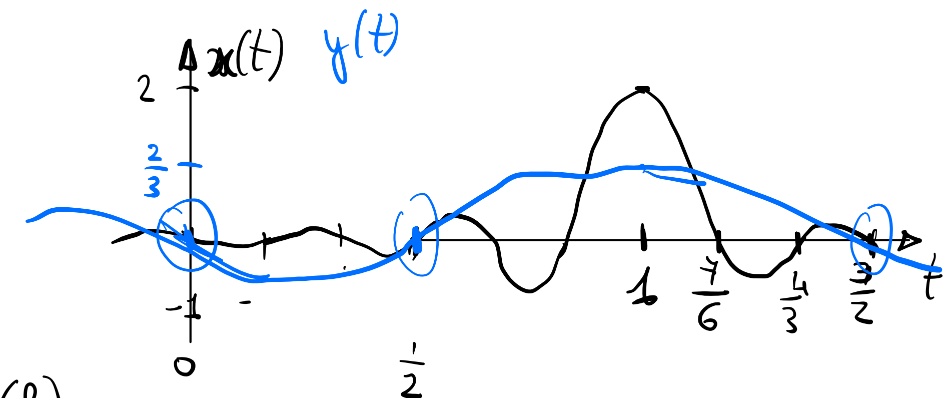


$$\underbrace{x(t) = 2\,\mathrm{sinc}\left[6(t-1)\right]}$$

$2\,\mathrm{sinc}(6t)$
$\downarrow$
$6(t-1)$

$H(f) = \mathrm{rect}\left(\frac{f}{2}\right)$

$x(t) \to$ $\boxed{H(f)}$ $\to y(t)$



$X(f)$: $\quad 2\,\mathrm{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\,\mathrm{rect}(f)$

$\downarrow$

$2\,\mathrm{sinc}(6t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{6}\mathrm{rect}\left(\frac{f}{6}\right)$

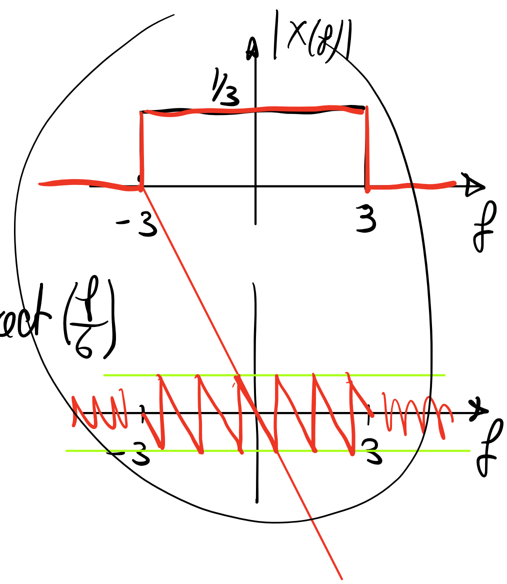
$\downarrow$

$2\,\mathrm{sinc}(6(t-1)) \longleftrightarrow \boxed{\frac{1}{3}\mathrm{rect}\left(\frac{f}{6}\right)e^{-j2\pi f} = X(f)}$

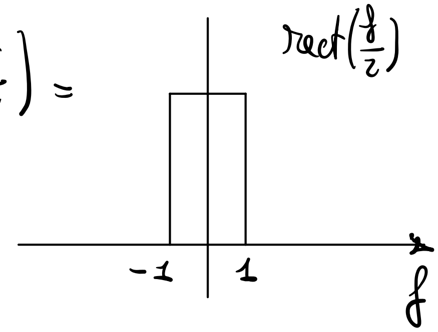
$$\to \left|X(f)\right| = \left|\frac{1}{3}\mathrm{rect}\left(\frac{f}{6}\right)e^{-j2\pi f}\right| = \frac{1}{3}\mathrm{rect}\left(\frac{f}{6}\right)\underbrace{\left|e^{-j2\pi f}\right|}_{1} = \frac{1}{3}\mathrm{rect}\left(\frac{f}{6}\right)$$

$$\measuredangle X(f) = \measuredangle\underbrace{\left(\frac{1}{3}\mathrm{rect}\left(\frac{f}{6}\right)\right)}_{0} + \measuredangle e^{-j2\pi f} = -2\pi f \;,\; |f|\leq 3$$

$\nexists$ altrove



$y(t)$? : $Y(f) = X(f)\,H(f) = \frac{1}{3}\,\mathrm{rect}\left(\frac{f}{6}\right) e^{-j2\pi f} \cdot \mathrm{rect}\left(\frac{f}{2}\right) =$



$$= \frac{1}{3}\,\mathrm{rect}\left(\frac{f}{2}\right) \underbrace{e^{-j2\pi f}}$$

$$\frac{1}{3}\,\mathrm{rect}(f) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{3}\,\mathrm{sinc}(t)$$

$f \to \frac{f}{2}$ $$\frac{1}{3}\,\mathrm{rect}\left(\frac{f}{2}\right) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{3}\,2\,\mathrm{sinc}(2t)$$

$$\frac{1}{3}\,\mathrm{rect}\left(\frac{f}{2}\right) e^{-j2\pi f} \xrightarrow{\mathcal{F}^{-1}} \boxed{\frac{2}{3}\,\mathrm{sinc}\left(2(t-1)\right) = y(t)}$$