Lecture 1: Graph minors

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A graph is a pair $G = (V, E)$ where $E \subseteq \binom{V}{2}$. We only consider finite graphs and write $n = |V(G)|$ and $m = |E(G)|$. With a graph G we often mean the isomorphism class of G (the set of all graphs isomorphic to G). Given a graph $G = (V, E)$ and $U \subseteq V$, we denote by $G[U] = (U, E \cap \binom{U}{2})$ the subgraph of G induced by U. If H is a subgraph of G (not necessarily induced) then we write $H \subseteq G$. The neighborhood of $v \in V$ is $N(v) = \{u \in V : \{u, v\} \in E\}$. For any $a, b \in \mathbb{N}$ let $[a, b] = \{a, \ldots, b\}$ and $[a] = [1, a]$.

Some important graphs are:

- the complete graph K_n, where $V(K_n) = [n]$ and $E(K_n) = \binom{n}{2}$
- the complete bipartite graph $K_{\ell,r}$, where $V(K_{\ell,r}) = L \dot\cup R$, $|L| = \ell$ and $|R| = r$, and $E = \{\{u, v\} : u \in L, v \in R\}$. Here $\dot\cup$ denotes disjoint union.
- P_n, the path on n vertices
- C_n, the cycle on n vertices

1 Graph minors

The following operations are defined on every graph $G = (V, E)$:

- deletion of a vertex $v \in V$: yields the graph $G \setminus v = G[V \setminus \{v\}]$
- deletion of an edge $e \in E$: yields the graph $G \setminus e = (V, E \setminus \{e\})$
- contraction of an edge $e = \{u, v\} \in E$: yields the graph $G/e = (V', E')$,

$$V' = V \setminus \{u, v\} \cup \{uv\} \tag{1}$$

$$E' = E \setminus \Big\{\{x, u\}, \{x, v\} : x \in V\Big\} \cup \Big\{\{uv, x\} : x \in N(u) \cup N(v) \setminus \{u, v\}\Big\} \tag{2}$$

Definition 1.1. A graph H is a *minor* of a graph G, written $H \preceq G$, if H can be obtained from G by a sequence of vertex deletions, edge deletions, and edge contractions. It is a *proper* minor if $H \neq G$.

Note that by H we mean any graph isomorphic to H (see notes above). Note also that minors and subgraphs are different.

Exercise 1. *Define the property of being a subgraph using the three operations above, then find H, G such that $H \preceq G$ but $H \not\subseteq G$. What about the vice versa? What about induced subgraphs?*

Exercise 2. *Is P_n a minor of K_n? Is K_n a minor of $K_{\ell,r}$? And vice versa?*

Exercise 3. *What are the minors of a tree? Of a forest? What graphs have a K_3 minor?*

A *graph property* is a set (or family, or class) of graphs that is closed under isomorphism. For instance, $\mathcal{F} = \{G : G \text{ is a forest}\}$ is the property of being acyclic. If H is not a minor of G then we say G is H-minor-free and write $H \not\preceq G$. A family of graphs $\mathcal{F}$ is *H-minor-free* if $H \not\preceq G$ for all $G \in \mathcal{F}$.

Exercise 4. *If $H \not\preceq G$, is also $H' \not\preceq G$ for every $H' \preceq H$? And vice versa?*

Lemma 1.2. *The family of acyclic graph is precisely the family of K_3-minor-free graphs.*

2 Basic properties

Prove that $\preceq$ is transitive, that is:

Lemma 2.1. *If $H \preceq G$ and $G \preceq F$ then $H \preceq F$.*

A *subdivision* of G is any graph obtained from G by replacing every edge with a nontrivial (i.e., with at least one edge) path. The ℓ-subdivision of G is the graph obtained by replacing every edge of G with a copy of $P_{\ell+1}$. For instance, the 0-subdivision of G is G itself.

Exercise 5. *Prove that if G is a subdivision of H then $G \succeq H$.*

Here is a first intuitive characterization of minors:

Lemma 2.2. *$H \preceq G$ iff H can be obtained by contracting edges of some $F \subseteq G$.*

Proof. If H can be obtained by contracting edges of $F \subseteq G$, then $H \preceq G$, since F is obtained by deleting vertices and edges of G.

Now suppose $H \preceq G$. Let $O = o_1, \ldots, o_\ell$ be the sequence of operations that produces H from G. If $O = O_1 O_2$ where O_1 is a sequence of deletions and O_2 a sequence of contractions, then H is a contraction of a subgraph of G and we are done. Suppose instead that there is $i \in [\ell - 1]$ such that o_i is a contraction of $e = \{u, v\}$ and o_{i+1} is a deletion. We define a sequence of operations O' according to o_{i+1} as follows:

- if o_{i+1} deletes a vertex $w \neq uv$ then obtain O' by switching o_i and o_{i+1}.
- if o_{i+1} deletes uv then obtain O' by replacing o_i, o_{i+1} with the deletion of u and v.
- if o_{i+1} deletes an edge $e' = \{x, y\} \not\ni uv$ then obtain O' by switching o_i and o_{i+1}.
- if o_{i+1} deletes an edge $e' = \{uv, y\}$ then obtain O' by replacing o_i, o_{i+1} with the deletion of every edge between y and $\{u, v\}$ followed by the contraction of e.

Observe that in any case O' is equivalent to O.

Now let $N(O)$ be the number of pairs (i, j) with $j > i$ such that o_i is a contraction and o_j is a deletion. Note that $N(O') < N(O)$; hence, repeating the construction above yields a sequence O^* equivalent to O such that $N(O^*) = 0$. But $N(O^*) = 0$ means $O^* = O_1^* O_2^*$, which by the observation above completes the proof. □

Definition 2.3. Let H and G be graphs. A *model* of H in G is a function $f : V(H) \to 2^{V(G)}$ such that:

1. $\forall u, v \in V(H)$, $u \neq v$, $f(u) \cap f(v) = 0$
2. $\forall v \in V(H)$, $G[f(v)]$ is connected
3. $\forall \{u, v\} \in E(H)$, in G there is an edge between $f(u)$ and $f(v)$

Here is an even more intuitive characterization of minors:

Lemma 2.4. *$H \preceq G$ if and only if there is a model of H in G.*

Proof. Suppose there is a model f of H in G. Delete all vertices in $V(G) \setminus \cup_{v \in V(H)} f(v)$, then delete all edges in $E(G)$ between any $f(u)$ and $f(v)$ such that $\{u, v\} \notin E(H)$. This yields a subgraph of G on $\cup_{v \in V(H)} f(v)$, and by contracting all edges of $G[f(v)]$ for all $v \in V(H)$, we obtain H. By Lemma 2.2 this implies $H \preceq G$.

Now suppose $H \preceq G$. By Lemma 2.2 H can be obtained by contracting edges of some $F \subseteq G$. Note that the set of edges contracted must forms a spanning forest of F in the form $\{T_u\}_{u \in V(H)}$; so that, for every $u \in V(H)$, contracting all edges of T_u yields u. Setting $f(u) = V(T_u)$ yields a model of H in G. □

By varying the set of operations allowed, we obtain variants of the notion of minor.

Definition 2.5. A graph H is an *induced minor* of a graph G, written $H \preceq G$, if H can be obtained from G by a sequence of vertex deletions and edge contractions, and it is a *topological* minor of G if it can be obtained from G by a sequence of edge contractions.

Exercise 6. *Adapt Lemma 2.4 for induced minors and topological minors.*

3 Hadwiger's Conjecture

Let G be a graph and $k \in \mathbb{N}$. A *k-coloring* of G is a function $c : V(G) \to [k]$. A coloring c is *proper* if $c(u) \neq c(v)$ for all $\{u, v\} \in E$.

Definition 3.1. The *chromatic number* $\chi(G)$ of a graph G is:

$$\chi(G) = \min\{k \in \mathbb{N} \,:\, G \text{ has a proper } k\text{-coloring}\} \tag{3}$$

Exercise 7. *Prove that $\chi(K_k) = k$ for all $k \in \mathbb{N}$.*

Exercise 8. *Prove that $\chi(G) \leq 2$ if and only if G is bipartite. More in general prove that $\chi(G) = k$ if and only if $V(G) = \dot\cup_{i \in [k]} V_i$ where V_i is an independent set of G for every $i \in [k]$.*

Definition 3.2. The *Hadwiger number* $h(G)$ of G is:

$$h(G) = \max\{k \in \mathbb{N} \,:\, K_k \preceq G\} \tag{4}$$

This is often referred to as the most important open problem in graph theory:

Conjecture 3.3 (Hadwiger's Conjecture, 1943)**.** *$\chi(G) \leq h(G)$ for every G.*

Exercise 9. *Show the existence of arbitrarily large graphs G with $\chi(G) = h(G)$, or with $\chi(G) = \mathcal{O}(1)$ and $h(G) = \Omega(\sqrt{n})$.*

Here are some cases of Hadwiger's conjecture for small $h(G)$:

- The conjecture holds when $h(G) = 1$. Indeed, $h(G) = 1$ means G has no edges, in which case G has a proper 1-coloring, so $\chi(G) \leq 1$.
- The conjecture holds for $h(G) = 2$. Indeed, $h(G) = 2$ means G is a forest, in which case G has a proper 2-coloring, so $\chi(G) \leq 2$.
- we know the conjecture holds for $3 \leq h(G) \leq 6$ as well; for $h(G) \geq 7$ we do not know

4 Forbidden minors

Some properties can be characterized by *forbidden minors*.

Definition 4.1. For any set of graphs $\mathcal{H}$ we define $\mathrm{Forb}(\mathcal{H}) = \{G : G \not\succeq H, \forall H \in \mathcal{H}\}$.

For instance:

Lemma 4.2. *The class of acyclic graphs is* $\mathrm{Forb}(\{K_3\})$.

The class $\mathcal{H}$ is called *obstruction set* or *Kuratowski set* for $\mathrm{Forb}\,\mathcal{H}$. Hence $\mathcal{H} = \{K_3\}$ is the obstruction set for acyclicity. Obstruction sets give us the "reason" behind a property, and a "certificate" that a given graph does not possess that property. We can prove that $\mathcal{H} = \{K_5, K_{3,3}\}$ is the obstruction set for planarity:

Theorem 4.3 (Wagner, 1937)**.** *The class of planar graphs is* $\mathrm{Forb}(K_5, K_{3,3})$.

Proof. We start by recalling Kuratowski's theorem: a graph G is planar if and only if $H \not\subseteq G$ whenever H is the subdivision of K_5 or $K_{3,3}$.

Suppose first G is nonplanar. Then by Kuratowski's theorem $H \subseteq G$, and thus $H \preceq G$, for some subdivision H of K_5 or $K_{3,3}$. But then H has K_5 or $K_{3,3}$ as minor, and by transitivity of $\preceq$ this holds for G as well.

Suppose instead G is planar. Observe that a planar graph has only planar minors. Indeed, vertex deletion and edge deletion clearly preserve planarity; just note that edge contraction preserves planarity, too. But K_5 and $K_{3,3}$ are nonplanar by Kuratowski's theorem; therefore $K_5, K_{3,3} \not\preceq G$. $\square$

Note that for $h(G) = 4$ Wagner's theorem and Hadwiger's conjecture imply the four-color theorem (which says that if G is planar then $\chi(G) \leq 4$). Indeed, if G is planar then by Wagner's theorem $h(G) \leq 4$, thus by Hadwiger's conjecture $\chi(G) \leq 4$.

Exercise 10. *Is is true that* $\mathrm{Forb}(K_4)$ *is the class of* 4*-colorable graphs?*