Linear programming exercises
Part 2

Giovanni Righini

Exercise 2.1: post-optimal analysis.

Given the following LP,

$$\begin{aligned} \text{maximize } z = & x_1 + 2x_2 \\ \text{s.t. } & x_2 \leq 2x_1 + 2 \\ & x_1 + 3x_2 \leq 27 \\ & x_1 + x_2 \leq 15 \\ & 2x_1 \leq x_2 + 18 \\ & x \geq 0 \end{aligned}$$

and its optimal tableau (see Exercise 1.1)

21	0	0	0	1/2	1/2	0
14	0	0	1	-3/2	7/2	0
6	0	0	0	3/2	-7/2	1
6	0	1	0	1/2	-1/2	0
9	1	0	0	-1/2	3/2	0

1. discuss the robustness of the optimal solution with respect to variations of the marginal revenues, interpreting z as a profit;
2. which resources can be scarce if c_1 can vary between $1/2$ and $3/2$?

Question 1: sensitivity analysis.
The optimal tableau corresponds to the optimal basis $B^* = \{1, 2, 3, 6\}$ and to the optimal basic solution $x^* = [9\ 6\ 14\ 0\ 0\ 6]$ with $z^* = 21$.

Variations of c_1. Examining the tableau at optimality,

21	0	0	0	1/2	1/2	0
6	0	1	0	1/2	-1/2	0
9	1	0	0	-1/2	3/2	0
14	0	0	1	-3/2	7/2	0
6	0	0	0	3/2	-7/2	1

since x_1 is basic on row 2, we have

$$\frac{-1/2}{3/2} \leq \Delta c_1 \leq \frac{-1/2}{-1/2}$$

that is

$$-\frac{1}{3} \leq \Delta c_1 \leq 1$$

which means

$$\frac{2}{3} \leq c_1 \leq 2.$$

Variations of c_2. Examining the tableau at optimality,

21	0	0	0	1/2	1/2	0
6	0	1	0	1/2	-1/2	0
9	1	0	0	-1/2	3/2	0
14	0	0	1	-3/2	7/2	0
6	0	0	0	3/2	-7/2	1

since x_2 is basic on row 1, we have

$$\frac{-1/2}{1/2} \leq \Delta c_2 \leq \frac{-1/2}{-1/2}$$

that is

$$-1 \leq \Delta c_2 \leq 1$$

which means

$$1 \leq c_2 \leq 3.$$

Question 2: parametric analysis.

At optimality (vertex D in the figure), the resources corresponding to x_4 and x_5 are scarce, since the corresponding constraints are active and x_4 and x_5 are non-basic. However, sensitivity analysis reveals that B^* remains optimal only for $\frac{2}{3} \leq c_1 \leq 2$. So, we have no information about scarce resources when $\frac{1}{2} \leq c_1 \leq \frac{2}{3}$. For this purpose we need parametric analysis on c_1.

We already know from sensitivity analysis that the ratio that bounds the allowable decrease of c_1 is $\frac{-1/2}{3/2}$ which is found on column 5. In other words x_5 becomes basic when c_1 decreases by more than $1/3$. When c_1 decreases by $1/3$, the indifference lines of the objective function become parallel to constraint (4) and we have two equivalent optimal solutions (vertices C and D in the figure). This corresponds to the occurrence of a zero reduced cost in the tableau.

We need to perform a pivot step so that column 5 leaves the basis, in order to explore what happens for $c_1 < 2/3$. Therefore, we need to reconstruct the tableau that would have been obtained in vertex D with $c_1 = 2/3$ instead of $c_1 = 1$. The only difference would have been in row 0, because the entries in the other rows of the tableau do not depend on the coefficients c.

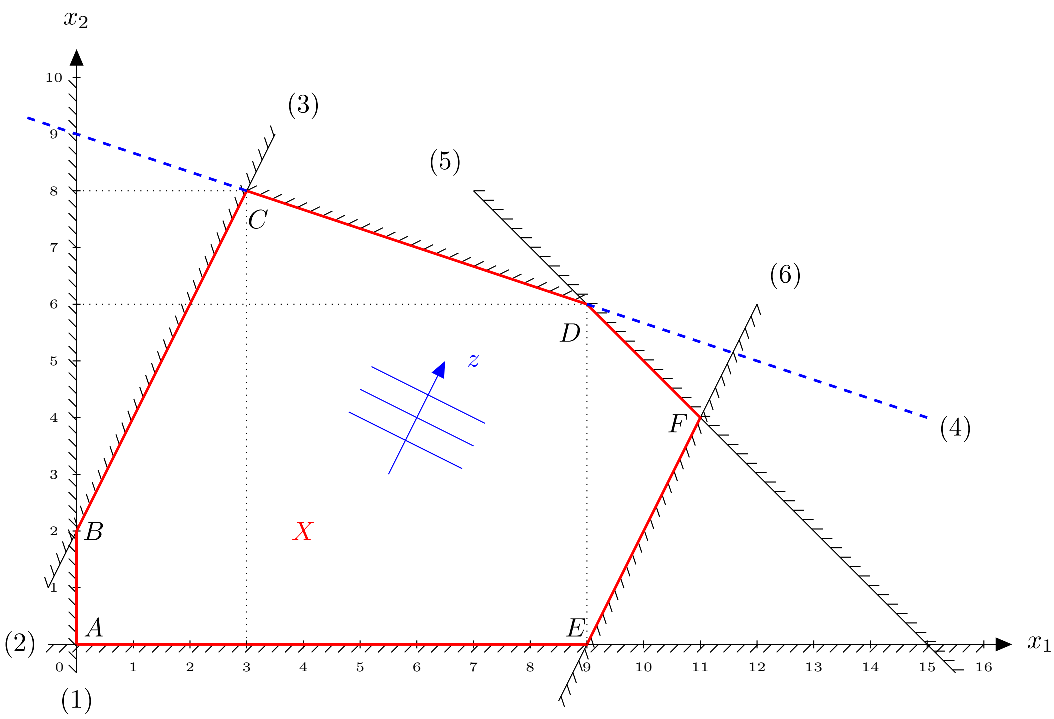
First of all, we have to remember that in standard form we put the objective funnction in minimization form. So, we are minimizing $z' = -c_1x_1 - 2x_2$. we can easily compute the value of z' in vertex D when $c_1 = 2/3$. We have $x_D = [9\ 6\ 14\ 0\ 0\ 6]$. Hence, $z'(D) = -9c_1 - 12$; For $c_1 = 1$, $z'(D) = -21$ and this is consistent with the result already found. for $c_1 = 2/3$, $z'(D) = -18$. Therefore in the top left corner of the tableau we would have obtained an entry equal to 18 instead of 21.

The reduced costs on the basic columns $\{1, 2, 3, 6\}$ would have been equal to 0, by definition of canonical form.

The reduced cost of the non-basic variables must be computed. We know that in a canonical form, $z' = z'_B + ({c'_N}^T - {c'_B}^T B^{-1}N)x_N$ and $Ix_B + B^{-1}Nx_N = B^{-1}b$. By c' we indicate the coefficents of z', which are opposite in sign to the coefficients of the original objective z. We can read the matrix $B^{-1}N$ from the current tableau, after reordering the rows in order to obtain an identity matrix in the basic columns:

$$B^{-1}N = \begin{bmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \\ -3/2 & 7/2 \\ 3/2 & -7/2 \end{bmatrix}.$$

Since $c'_B = [-c_1\ -2\ 0\ 0]^T$ and $c'_N = [0\ 0]^T$, we obtain $z'(D) = z'_B + (-1/2\, c_1 + 1)x_4 + (3/2\, c_1 - 1)x_5$. When $c_1 = 1$, both reduced costs of x_4 and x_5 are equal to $1/2$, and this is consistent with the tableau in vertex D we have obtained with the simplex algorithm. When $c_1 = 2/3$ the same formula gives us the reduced costs $2/3$ for x_4 and 0 (as expected) for x_5. So, pivoting on column 5 does not change the value of z'.



Therefore for $c_1 = 2/3$, the tableau in vertex D reads as follows:

18	0	0	0	2/3	0	0
6	0	1	0	1/2	-1/2	0
9	1	0	0	-1/2	3/2	0
14	0	0	1	-3/2	**7/2**	0
6	0	0	0	3/2	-7/2	1

Now we can pivot on column 5 to reach a new equivalent basic solution.

To pivot on column 5 keeping the basis feasible, the pivot must be chosen according to the usual rules: there are two candidate pivots on column 5, one on row 2 and the other on row 3. The minimum ratio is $\frac{14}{7/2}$, which is smaller than $\frac{9}{3/2}$. Therefore the pivot in on column 5, row 3 (in bold). The variable leaving the basis is thus x_3, which is basic on row 3. The starting tableau is on the left, the resulting tableau is on the right.

18	0	0	0	2/3	0	0
6	0	1	0	1/2	-1/2	0
9	1	0	0	-1/2	3/2	0
14	0	0	1	-3/2	**7/2**	0
6	0	0	0	3/2	-7/2	1

18	0	0	0	2/3	0	0
8	0	1	1/7	2/7	0	0
3	1	0	-3/7	1/7	0	0
4	0	0	2/7	-3/7	1	0
20	0	0	1	0	0	1

$B = \{1, 2, 3, 6\}$ $\quad x = [9\ 6\ 14\ 0\ 0\ 6]$ $\quad z = 21$

$B = \{1, 2, 5, 6\}$ $\quad x = [3\ 8\ 0\ 0\ 4\ 20]$ $\quad z = 18$

Now it is possible to repeat the sensitivity analysis on c_1 around the new basic solution. Variable x_1 is still basic on row 2. A lower bound for Δc_1 is given by $\frac{-2/3}{1/7}$, i.e. $-14/3$. For $\Delta c_1 = -14/3$, we have $c_1 = 2/3 - 14/3 = -4$, which includes the required range $1/2 \leq c_1 \leq 3/2$.

Therefore vertex C remains optimal for $1/2 \leq c_1 \leq 2/3$ and this concludes the required parametric analysis on c_1.

Exercise 2.2: post-optimal analysis.

Given the following LP,

$$\begin{aligned} \text{maximize } z =& 2x_1 + 3x_2 + 4x_3 + 5x_4 \\ \text{s.t. } & x_1 + x_2 - x_3 + x_4 \leq 10 \\ & x_1 + 2x_2 \leq 8 \\ & x_3 + x_4 \leq 20 \\ & x \geq 0 \end{aligned}$$

and its optimal tableau (see exercise 1.2),

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

assuming z is the profit of a manufacturing company and b is the amount of available resources, whose current price is $1/2$, 1 and 2, answer these questions with post-optimal analysis.

1. An offer is issued by a provider for an additional amount of the third resource at a price equal to 4. Is it profitable to accept it? What amount of resource should be purchased?
2. Which of the three resources is subject to the largest increase in value due to its transformation in the manufacturing plant?
3. What is the maximum amount of the first resource that could be profitably used, if available at negligible price?
4. Do the sensitivity analysis on all coefficients of the objective function and all right-hand-sides of the constraints.

Question 1.

All three constraints are active at optimality. In particular, the slack variable x_7, corresponding to the third resource, has reduced cost $9/2$; i.e. the shadow price of the third resource is $9/2$. Hence, although the price 4 is definitely larger than the price of the usual provision of resource (at a price $1/2$) it is still convenient to accept it, because its shadow price is larger than its price. This remains true in the range in which the optimal basis does not change. Examining the optimal tableau, and in particular column 7, we see that the increase of its right hand side is not bounded. Therefore the shadow price remains equal to $9/2$ for any additional quantity of resource. Hence it is always profitable to buy at price 4 any available amount of the resource.

Question 2.

The increase in value of the three resources can be immediately obtained by comparing the price at which they are purchased and their shadow price, i.e,. their actual value for the company.

First resource: $\frac{1/2-1/2}{1/2} = 0\%$.
Second resource: $\frac{3/2-1}{1} = 50\%$.
Third resource: $\frac{9/2-2}{2} = 125\%$.

Question 3.

The answer is given by the value of the right hand side beyond which the resource becomes non-scarce and its corresponding constraint becomes non-active. The first resource corresponds to the non-basic variable x_5. From the sensitivity analysis on column 5, we see that

$$\frac{-11}{1/2} \leq \Delta b \leq \frac{-9}{-1/2}$$

i.e.

$$-22 \leq \Delta b \leq 18.$$

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

This guarantees that at least 18 additional units of resource would be used, if available. Now we need to know whether x_5 would remain non-basic if $\Delta b > 18$; parametric analysis provides the answer. Pivoting on column 5, row 3, i.e. on the element defining the allowable increase, we obtain the following tableau.

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

$$B = \{1, 3, 4\} \quad x = [8\ 0\ 9\ 11\ 0\ 0\ 0] \quad z = 107$$

116	0	1	1	0	0	2	5
20	0	0	1	1	0	0	1
8	1	2	0	0	0	1	0
-18	0	-1	-2	0	1	-1	-1

$$B = \{1, 4, 5\} \quad x = [8\ 0\ 0\ 20\ -18\ 0\ 0] \quad z = 116$$

The solution is infeasible because we have pivoted on a negative coefficient, moving beyond constraint (5). Now we shift the constraint, so that it passes through the current basic solution, by replacing the entry -18 in column 0 with 0.

116	0	1	1	0	0	2	5
20	0	0	1	1	0	0	1
8	1	2	0	0	0	1	0
0	0	-1	-2	0	1	-1	-1

Now the current solution is degenerate. To go on, we have to make x_5 non-basic again. However this is not possible, because there are no available candidate pivots on row 3. This means that beyond this value, x_5 would remain basic: the optimal solution is now determined by the other constraints and it would not change even if constraint (5) were moved further. Hence, it is not profitable to buy more than 18 additional units of the first resource.

Question 4.

Sensitivity analysis on c_1.

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

Column 1 is basic on row 2.

$$\max\left\{\frac{-1/2}{2}, \frac{-3/2}{1}\right\} \leq \Delta c_1 < \infty$$

$$-1/4 \leq \Delta c_1 < \infty$$

Sensitivity analysis on c_2.

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

Column 2 is non-basic. Then

$$\Delta c_2 \leq 1/2.$$

Sensitivity analysis on c_3.

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

Column 3 is basic on row 3.

$$\max\left\{\frac{-1/2}{1/2}, \frac{-1/2}{1/2}, \frac{-9/2}{1/2}\right\} \leq \Delta c_1 \leq \frac{-1/2}{-1/2}$$

$$-1 \leq \Delta c_3 \leq 1$$

Sensitivity analysis on b_1.

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

Row 1 corresponds to slack variable x_4, which is non-basic.

$$\frac{-11}{1/2} \leq \Delta b_1 \leq \frac{-9}{-1/2}$$

$$-22 \leq \Delta b_1 \leq 18.$$

Sensitivity analysis on b_2.

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

Row 2 corresponds to slack variable x_5, which is non-basic.

$$\max\left\{\frac{-8}{1}, \frac{-9}{1/2}\right\} \leq \Delta b_2 \leq \frac{-11}{-1/2}$$

$$-8 \leq \Delta b_2 \leq 22.$$

Sensitivity analysis on b_3.

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

Row 2 corresponds to slack variable x_5, which is non-basic.

$$\max\left\{\frac{-11}{1/2}, \frac{-9}{1/2}\right\} \leq \Delta b_3 < \infty$$

$$-18 \leq \Delta b_3 < \infty.$$

Exercise 2.3: duality.

Given the following LP,

$$\begin{aligned} \text{maximize } z = & x_2 \\ \text{s.t. } & x_1 - 2x_2 \leq -2 \\ & -2x_1 + x_2 \leq -4 \\ & x_1 + x_2 \leq 4 \\ & x \geq 0 \end{aligned}$$

1. write its dual;
2. solve the dual with the simplex algorithm;
3. solve the primal geometrically.

Question 1.

$$\begin{aligned} \text{maximize } z = & x_2 \\ \text{s.t. } & x_1 - 2x_2 \leq -2 \\ & -2x_1 + x_2 \leq -4 \\ & x_1 + x_2 \leq 4 \\ & x \geq 0 \end{aligned} \qquad \begin{aligned} \text{minimize } w = & -2y_3 - 4y_4 + 4y_5 \\ \text{s.t. } & y_3 - 2y_4 + y_5 \geq 0 \\ & -2y_3 + y_4 + y_5 \geq 1 \\ & y \geq 0 \end{aligned}$$

Question 2.

The initial basis is infeasible. We define an auxiliary problem, where the violated constraint temporarily plays the role of the objective function.

0	0	0	-2	-4	4
0	1	0	-1	2	-1
-1	0	1	2	-1	-1

$B = \{1, 2\}$ $y = [0\ -1\ 0\ 0\ 0]$ $w = 0$

-1	0	1	2	-1	-1
0	1	0	-1	2	-1
0	0	0	-2	-4	4

Iteration 1. We can pivot on column 4 or column 5. Selecting column 4, the following pivot step is done.

-1	0	1	2	-1	-1
0	1	0	-1	2	-1
0	0	0	-2	-4	4

$B = \{1, 2\}$ $y = [0\ -1\ 0\ 0\ 0]$ $w = 0$

-1	1/2	1	3/2	0	-3/2
0	1/2	0	-1/2	1	-1/2
0	2	0	-4	0	2

$B = \{2, 4\}$ $y = [0\ -1\ 0\ 0\ 0]$ $w = 0$

The constraint is still violated.

Iteration 2. We can pivot on column 5. Since there no positive candidate pivots on column 5, the pivot must be selected on the row of the violated constraint (the auxiliary problem is unbounded).

-1	1/2	1	3/2	0	-3/2
0	1/2	0	-1/2	1	-1/2
0	2	0	-4	0	2

$B = \{2, 4\}$ $y = [0\ -1\ 0\ 0\ 0]$ $w = 0$

2/3	-1/3	-2/3	-1	0	1
1/3	1/3	-1/3	-1	1	0
-4/3	8/3	4/3	-2	0	0

$B = \{4, 5\}$ $y = [0\ 0\ 0\ 1/3\ 2/3]$ $w = 4/3$

Now the basis is feasible. The tableau of the original dual problem can be reconstructed.

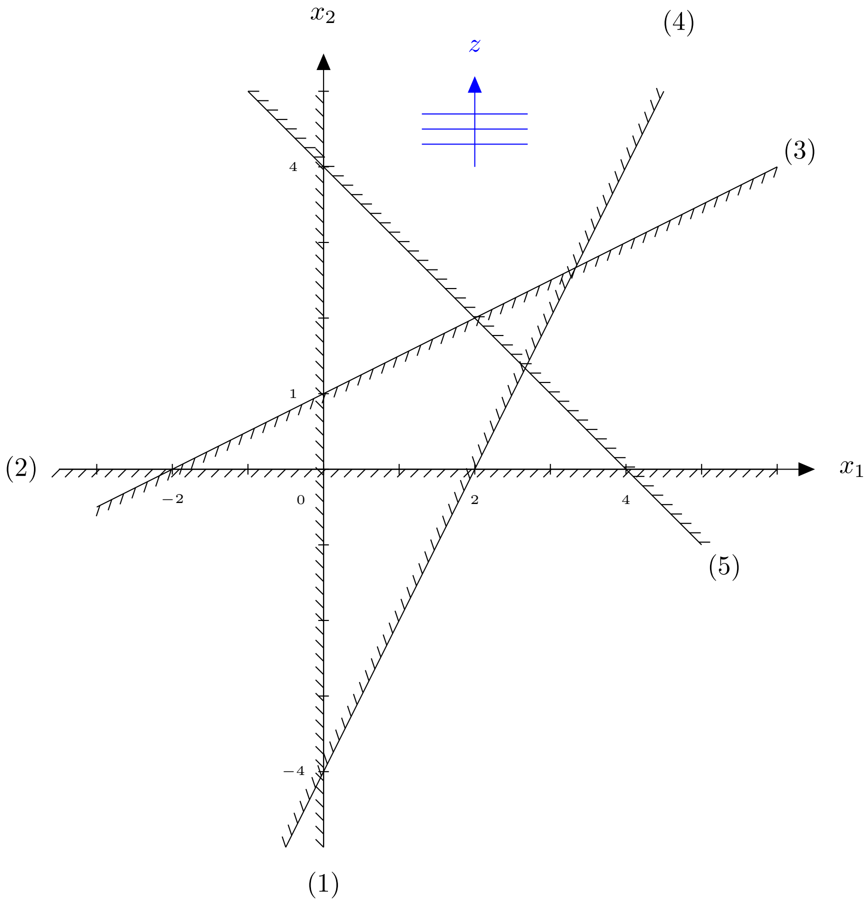
-4/3	8/3	4/3	-2	0	0
1/3	1/3	-1/3	-1	1	0
2/3	-1/3	-2/3	-1	0	1

$B = \{4, 5\}$ $y = [0\ 0\ 0\ 1/3\ 2/3]$ $w = 4/3$

Column 3 is made by negative entries: the dual problem is unbounded. Hence, the primal problem is infeasible.

Question 3.

Since the primal problem has two variables, we can solve it by geometrical means.



The feasible region is empty; the primal problem is infeasible.

Exercise 2.4: duality.

Given the unbounded LP (see exercise 1.4),

$$\begin{aligned} \text{maximize } z = & x_1 + x_2 \\ \text{s.t. } & x_1 - x_2 \geq -2 \\ & -x_1 + 2x_2 \geq -1 \\ & x \geq 0 \end{aligned}$$

1. write its dual;
2. solve the dual with the simplex algorithm;
3. solve the dual geometrically.

Question 1.

$$\begin{aligned} \text{maximize } z = & x_1 + x_2 \\ \text{s.t. } & -x_1 + x_2 \leq 2 \\ & x_1 - 2x_2 \leq 1 \\ & x \geq 0 \end{aligned} \qquad \begin{aligned} \text{minimize } w = & 2y_3 + y_4 \\ \text{s.t. } & -y_3 + y_4 \geq 1 \\ & y_3 - 2y_4 \geq 1 \\ & y \geq 0 \end{aligned}$$

Question 2.

The initial basis is infeasible: both constraints are violated. We define an auxiliary problem, where constraint (1) temporarily plays the role of the objective function.

0	0	0	2	1
-1	1	0	1	-1
-1	0	1	-1	2

$B = \{4, 5\}$ $y = [0\,0\,0\,0\,-1]$ $w = 0$

-1	1	0	1	-1
-1	0	1	-1	2
0	0	0	2	1

Iteration 1. We pivot on column 4. The only available pivot is on the row of the violated constraint.

-1	1	0	1	-1
-1	0	1	-1	2
0	0	0	2	1

1	-1	0	-1	1
-3	2	1	1	0
-1	1	0	3	0

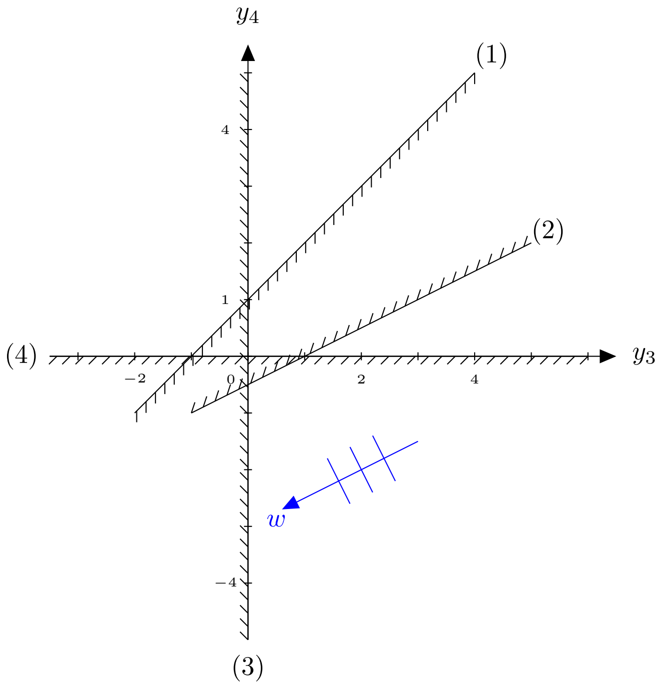
Feasibility with respect to constraint (1) has been repaired. We can now define a second auxiliary problem, using constraint (2) as a temporary objective function.

-3	2	1	1	0
1	-1	0	-1	1
-1	1	0	3	0

The violated constraint cannot be repaired, since all entries on row 0 are non-negative and the right-hand-side is negative. Hence the dual problem is infeasible.

Question 3.

Since the dual problem has two variables, we can solve it by geometrical means.



The feasible region is empty; the primal problem is infeasible.

Exercise 2.5: duality.

Given the following LP,

$$\begin{aligned} \text{maximize } z = \quad & 2x_1 + 6x_2 \\ \text{s.t. } \quad & -5x_1 + 2x_2 \leq 4 \\ & \qquad\quad 4x_2 \leq -3 \\ & x \geq 0 \end{aligned}$$

solve it and its dual.

Solution.

Primal problem	Dual problem
$\begin{aligned} \text{maximize } z = \quad & 2x_1 + 6x_2 \\ \text{s.t. } \quad & -5x_1 + 2x_2 \leq 4 \\ & \qquad\quad 4x_2 \leq -3 \\ & x \geq 0 \end{aligned}$	$\begin{aligned} \text{minimize } w = \quad & 4y_3 - 3y_4 \\ \text{s.t. } \quad & -5y_3 \qquad \geq 2 \\ & 2y_3 + 4y_4 \geq 6 \\ & y \geq 0 \end{aligned}$

The primal problem is infeasible, owing to the constraint $4x_2 \leq -3$.
The dual problem is infeasible too, owing to the constraint $-5y_3 \geq 2$.

Exercise 2.6: complementary slackness.

Given the following LP (see exercise 1.6),

$$\begin{aligned}\text{maximize } z = & x_1 + 2x_2 \\ \text{s.t. } & x_2 \leq 2x_1 + 2 \\ & x_2 \leq x_1 + 2 \\ & x_2 \leq \frac{1}{2}x_1 + 2 \\ & x_1 \leq 4 \\ & x \geq 0\end{aligned}$$

1. write its dual;
2. solve the dual with the dual simplex algorithm;
3. obtain the optimal solution of the dual from the optimal solution of the primal computed in exercise 1.6.

Question 1.

Primal problem

$$\begin{aligned}\text{maximize } z = & x_1 + 2x_2 \\ \text{s.t. } & x_2 \leq 2x_1 + 2 \\ & x_2 \leq x_1 + 2 \\ & x_2 \leq \frac{1}{2}x_1 + 2 \\ & x_1 \leq 4 \\ & x \geq 0\end{aligned}$$

Dual problem

$$\begin{aligned}\text{minimize } w = & 2y_3 + 2y_4 + 4y_5 + 4y_6 \\ \text{s.t. } & -2y_3 - y_4 - y_5 + y_6 \geq 1 \\ & y_3 + y_4 + 2y_5 \geq 2 \\ & y \geq 0\end{aligned}$$

Question 2.

In the dual tableau, feasibility conditions are violated (two constraints have negative right-hand-side), but optimality conditions are satisfied (all reduced costs are non-negative). Therefore, instead of having recourse to the initialization phase of the simplex algorithm to achieve feasibility, it is possible to start the dual simplex algorithm, with no initialization.

Iteration 1. We select the row corresponding to the lasgest violation of a constraint, i.e. row 2. There are three possible equivalent choices for the pivot column. As a tie-break rule, we choose the column with smallest index. Therefore we pivot on column 3.

0	0	0	2	2	4	4
-1	1	0	2	1	1	-1
-2	0	1	**-1**	-1	-2	0

$B = \{1, 2\}$
$y = [-1\ -2\ 0\ 0\ 0\ 0]$
$w = 0$

-4	0	2	0	0	0	4
-5	1	2	0	-1	-3	-1
2	0	-1	1	1	2	0

$B = \{1, 3\}$
$y = [-5\ 0\ 2\ 0\ 0\ 0]$
$w = 4$

Iteration 2. We observe that two reduced costs on columns 4 and 5 are null, even if the columns are non-basic. We pivot on row 1 and as a tie-break rule between column 4 and 5 we select the one with smallest index.

-4	0	2	0	0	0	4
-5	1	2	0	-1	-3	-1
2	0	-1	1	1	2	0

$B = \{1, 3\}$
$y = [-5\ 0\ 2\ 0\ 0\ 0]$
$w = 4$

-4	0	2	0	0	0	4
5	-1	-2	0	1	3	1
-3	1	1	1	0	-1	-1

$B = \{3, 4\}$
$y = [0\ 0\ -3\ 5\ 0\ 0]$
$w = 4$

In this iteration the basic solution has changed but the value of the objective function has not. Where the primal problem is degenerate, its dual has multiple equivalent solutions.

Iteration 3.

-4	0	2	0	0	0	4
5	-1	-2	0	1	3	1
-3	1	1	1	0	-1	-1

$B = \{3, 4\}$
$y = [0\ 0\ -3\ 5\ 0\ 0]$
$w = 4$

-4	0	2	0	0	0	4
-4	2	1	3	1	0	-2
3	-1	-1	-1	0	1	1

$B = \{4, 5\}$
$y = [0\ 0\ 0\ -4\ 3\ 0]$
$w = 4$

Again, a different basic solution but with the same objective value.
Iteration 4.

-4	0	2	0	0	0	4
-4	2	1	3	1	0	-2
3	-1	-1	-1	0	1	1

$B = \{4, 5\}$
$y = [0\ 0\ 0\ -4\ 3\ 0]$
$w = 4$

-12	4	4	6	2	0	0
2	-1	-1/2	-3/2	-1/2	0	1
1	0	-1/2	1/2	1/2	1	0

$B = \{5, 6\}$
$y = [0\ 0\ 0\ 0\ 1\ 2]$
$w = 12$

The solution is now feasible and optimal.

Question 3.

The optimal solution of the primal problem (see exercise 1.6) is

$$B = \{1, 2, 3, 4\} \quad x = [4\ 4\ 6\ 2\ 0\ 0] \quad z = 12$$

By the complementary slackness theorem, we know that optimality implies

$$y_1 = y_2 = y_3 = y_4 = 0.$$

Hence the dual problem reduces to

$$\begin{aligned} w =& 4y_5 + 4y_6 \\ \text{s.t.} & -y_5 + y_6 = 1 \\ & 2y_5 = 2 \end{aligned}$$

Solving this linear system of two equations with two variables, we obtain $y_5 = 1$, $y_6 = 2$ and $w = 12$.

Exercise 8: dual simplex algorithm.

Solve the following LP with the dual simplex algorithm and verify the optimal solution geometrically.

$$\begin{aligned}\text{minimize } z =& 8x_1 + 10x_2 + 24x_3\\ \text{s.t. } & -x_1 + x_2 + 3x_3 \geq 1\\ & 2x_1 + x_2 + x_3 \geq 2\\ & x \geq 0\end{aligned}$$

Question 1.

Iteration 1.

0	8	10	24	0	0
-1	1	-1	-3	1	0
-2	-2	-1	-1	0	1

$B = \{4, 5\}$
$x = [0\ 0\ 0\ -1\ -2]$
$z = 0$

-8	0	6	20	0	4
-2	0	-3/2	-7/2	1	1/2
1	1	1/2	1/2	0	-1/2

$B = \{1, 4\}$
$x = [1\ 0\ 0\ -2\ 0]$
$z = 8$

Iteration 2.

-8	0	6	20	0	4
-2	0	-3/2	-7/2	1	1/2
1	1	1/2	1/2	0	-1/2

$B = \{1, 4\}$
$x = [1\ 0\ 0\ -2\ 0]$
$z = 8$

-16	0	0	6	-4	6
4/3	0	1	7/3	-2/3	-1/3
1/3	1	0	-2/3	1/3	-1/3

$B = \{1, 2\}$
$x = [1/3\ 4/3\ 0\ 0\ 0]$
$z = 16$

Question 2.

The dual problem is as follows.

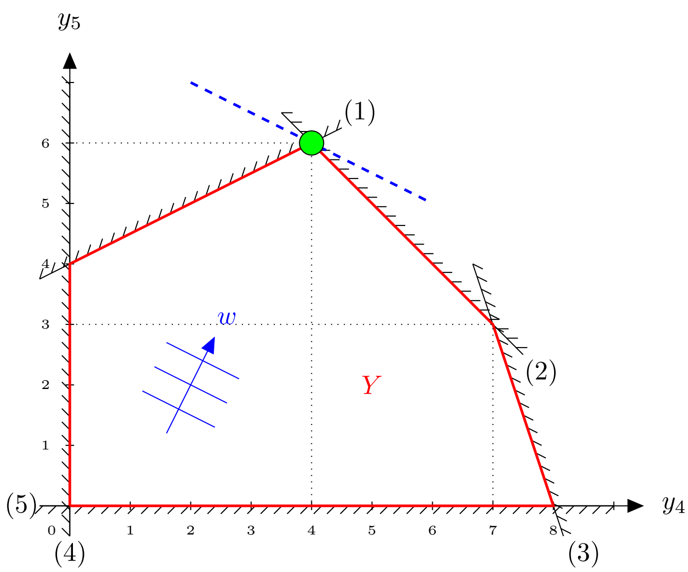
Primal problem

$$\begin{aligned}\text{minimize } z =& 8x_1 + 10x_2 + 24x_3\\ \text{s.t. } & -x_1 + x_2 + 3x_3 \geq 1\\ & 2x_1 + x_2 + x_3 \geq 2\\ & x \geq 0\end{aligned}$$

Dual problem

$$\begin{aligned}\text{minimize } w =& y_4 + 2y_5\\ \text{s.t. } & -y_4 + 2y_5 \leq 8\\ & y_4 + y_5 \leq 10\\ & 3y_4 + y_5 \leq 24\\ & y \geq 0\end{aligned}$$

Since the dual problem has only two variables, we can solve it geometrically. The geometrical representation of the dual problem is the following.


y_5
y_4
(1)
(2)
(3)
(4)
(5)
w
Y
0
1
2
3
4
5
6
7
8