Linear programming exercises
Part 1

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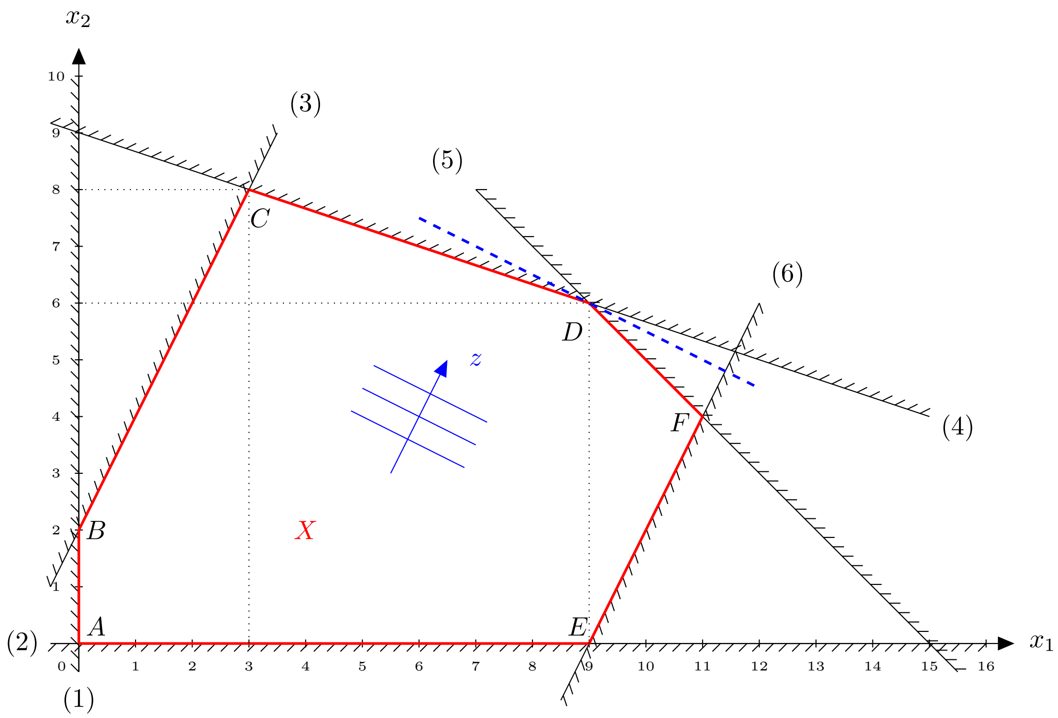
Exercise 1.1: finite optimal solution (2 dimensions).

Given the following LP,

$$
\begin{aligned}
\text{maximize } z =& x_1 + 2x_2 \\
\text{s.t. } & x_2 \leq 2x_1 + 2 \\
& x_1 + 3x_2 \leq 27 \\
& x_1 + x_2 \leq 15 \\
& 2x_1 \leq x_2 + 18 \\
& x \geq 0
\end{aligned}
$$

compute its optimal solution by geometrical arguments and with the simplex algorithm.

Geometrical solution.
By geometrical means, we obtain the following representation of the LP model.



The optimal solution is in $D = (9, 6)$ at the intersection of the constraints corresponding to the non-basic variables x_4 and x_5.

Solution with the simplex algorithm.
To solve the LP with the simplex algorithm, first of all we rewrite the model as a system of linear inequalities (maximization form and $\leq$ inequalities).

$$
\begin{aligned}
\text{maximize } z =& x_1 + 2x_2 \\
\text{s.t. } & -2x_1 + x_2 \leq 2 \\
& x_1 + 3x_2 \leq 27 \\
& x_1 + x_2 \leq 15 \\
& 2x_1 - x_2 \leq 18 \\
& x \geq 0
\end{aligned}
$$

Then, we put the problem in standard form by inserting four slack variables and replacing z with $w = -z$.

$$
\begin{aligned}
\text{minimize } w = & -x_1 - 2x_2 \\
\text{s.t. } & -2x_1 + x_2 + x_3 = 2 \\
& x_1 + 3x_2 + x_4 = 27 \\
& x_1 + x_2 + x_5 = 15 \\
& 2x_1 - x_2 + x_6 = 18 \\
& x \geq 0
\end{aligned}
$$

The corresponding tableau is the following.

0	-1	-2	0	0	0	0
2	-2	1	1	0	0	0
27	1	3	0	1	0	0
15	1	1	0	0	1	0
18	2	-1	0	0	0	1

It corresponds to a (strong) canonical form, where the basis is made by the last four columns, corresponding to the slack variables.

The current solution is in A. Active constraints are (1) and (2).

Column selection policy: first column with negative reduced cost.

Iteration 1. According to the numbering of the columns, the first one with negative reduced cost is column 1. To correctly select the pivot row, we discard row 1 (negative entry) and we compare the three ratios $27/1$ (row 2), $15/1$ (row 3) and $18/2$ (row 4). The minimum ratio is on row 4. Therefore we pivot on row 4, column 1, as indicated here below on the left, and we obtain the tableau on the right.

0	-1	-2	0	0	0	0
2	-2	1	1	0	0	0
27	1	3	0	1	0	0
15	1	1	0	0	1	0
18	**2**	-1	0	0	0	1

9	0	-5/2	0	0	0	1/2
20	0	0	1	0	0	1
18	0	7/2	0	1	0	-1/2
6	0	3/2	0	0	1	-1/2
9	1	-1/2	0	0	0	1/2

$B = \{3, 4, 5, 6\}$ $x = [0\ 0\ 2\ 27\ 15\ 18]$ $w = 0$ $B = \{1, 3, 4, 5\}$ $x = [9\ 0\ 20\ 18\ 6\ 0]$ $w = -9$

The current solution is in E. The active constraints are (2) and (6).

Let examine this pivot step in greater detail.

First we divide the pivot row, i.e. row 4 by the pivot, i.e. by 2. So, we compute the new row 4 as $(1/2)[\,18 \mid 2\ -1\ 0\ 0\ 0\ 1\,]$. The result is $[\,9 \mid 1\ -1/2\ 0\ 0\ 0\ 1/2\,]$.

Now we compute the rest of the tableau row by row.

Row 0: the entry of the tableau on row 0, column 1 (the pivot column) is -1. Then we subtract from row 0 the new pivot row multiplied by -1. So, we compute the new row 0 as $[\,0 \mid -1\ -2\ 0\ 0\ 0\ 0\,] - (-1)[\,9 \mid 1\ -1/2\ 0\ 0\ 0\ 1/2\,]$. The result is $[\,9 \mid 0\ -5/2\ 0\ 0\ 0\ 1/2\,]$.

Row 1: the entry of the tableau on row 1, column 1 (the pivot column) is -2. Then we subtract from row 1 the new pivot row multiplied by (-2). So, we compute the new row 1 as $[\,2 \mid -2\ 1\ 1\ 0\ 0\ 0\,] - (-2)[\,9 \mid 1\ -1/2\ 0\ 0\ 0\ 1/2\,]$. The result is $[\,20 \mid 0\ 0\ 1\ 0\ 0\ 1\,]$.

Row 2: the entry of the tableau on row 2, column 1 (the pivot column) is 1. Then we subtract from row 2 the new pivot row multiplied by 1. So, we compute the new row 2 as $[\,27 \mid 1\ 3\ 0\ 1\ 0\ 0\,] - 1[\,9 \mid 1\ -1/2\ 0\ 0\ 0\ 1/2\,]$. The result is

[18 | 0 7/2 0 1 0 $-1/2$].

Row 3: the entry of the tableau on row 3, column 1 (the pivot column) is 1. Then we subtract from row 3 the new pivot row multiplied by 1. So, we compute the new row 3 as [15 | 1 1 0 0 1 0] $-$ 1[9 | 1 $-1/2$ 0 0 0 1/2]. The result is [6 | 0 3/2 0 0 1 $-1/2$].

Iteration 2. Now the only column with negative reduced cost is column 2. There are only two candidate rows for pivoting: row 2 and row 3. Comparing the ratios $18/(7/2)$ and $6/(3/2)$, we select row 3. The tableaux before and after the pivot operation are shown here below.

9	0	-5/2	0	0	0	1/2
20	0	0	1	0	0	1
18	0	7/2	0	1	0	-1/2
6	0	**3/2**	0	0	1	-1/2
9	1	-1/2	0	0	0	1/2

19	0	0	0	0	5/3	-1/3
20	0	0	1	0	0	1
4	0	0	0	1	-7/3	2/3
4	0	1	0	0	2/3	-1/3
11	1	0	0	0	1/3	1/3

$B = \{1, 3, 4, 5\}$ $x = [9\ 0\ 20\ 18\ 6\ 0]$ $w = -9$ $\qquad$ $B = \{1, 2, 3, 4\}$ $x = [11\ 4\ 20\ 4\ 0\ 0]$ $w = -19$

The current solution is in F. The active constraints are (5) and (6).

Iteration 3. The only column with negative reduced cost is column 6. There are three candidate rows for pivoting: rows 1, 2 and 4. Comparing the ratios $20/1$, $4/(2/3)$ and $11/(1/3)$, we select row 2. The tableaux before and after the pivot operation are shown here below.

19	0	0	0	0	5/3	-1/3
20	0	0	1	0	0	1
4	0	0	0	1	-7/3	2/3
4	0	1	0	0	2/3	-1/3
11	1	0	0	0	1/3	1/3

21	0	0	0	1/2	1/2	0
14	0	0	1	-3/2	7/2	0
6	0	0	0	3/2	-7/2	1
6	0	1	0	1/2	-1/2	0
9	1	0	0	-1/2	3/2	0

$B = \{1, 2, 3, 4\}$ $x = [11\ 4\ 20\ 4\ 0\ 0]$ $w = -19$ $\qquad$ $B = \{1, 2, 3, 6\}$ $x = [9\ 6\ 14\ 0\ 0\ 6]$ $w = -21$

The current solution is in D. The active constraints are (4) and (5).

The algorithm stops, because optimality conditions are satified: all reduced costs on row 0 are non-negative.

Column selection policy: minimum reduced cost.

Iteration 1. Choosing a column with minimum reduced cost, we pivot on column 2. To correctly select the row, we discard row 4 (negative pivot) and we compare the three ratios $2/1$ (row 1), $27/3$ (row 2) and $15/1$ (row 3). The minimum ratio is that on row 1. Therefore we pivot on row 1, column 2, as indicated here below on the left, and we obtain the tableau on the right.

0	-1	-2	0	0	0	0
2	-2	**1**	1	0	0	0
27	1	3	0	1	0	0
15	1	1	0	0	1	0
18	2	-1	0	0	0	1

4	-5	0	2	0	0	0
2	-2	1	1	0	0	0
21	7	0	-3	1	0	0
13	3	0	-1	0	1	0
20	0	0	1	0	0	1

$B = \{3, 4, 5, 6\}$
$x = [0\ 0\ 2\ 27\ 15\ 18]$
$w = 0$

$B = \{2, 4, 5, 6\}$
$x = [0\ 2\ 0\ 21\ 13\ 20]$
$w = -4$

The current solution is in B. The active constraints are (1) and (3).

Iteration 2.

4	-5	0	2	0	0	0
2	-2	1	1	0	0	0
21	**7**	0	-3	1	0	0
13	3	0	-1	0	1	0
20	0	0	1	0	0	1

$B = \{2, 4, 5, 6\}$
$x = [0\ 2\ 0\ 21\ 13\ 20]$
$w = -4$

19	0	0	-1/7	5/7	0	0
8	0	1	1/7	2/7	0	0
3	1	0	-3/7	1/7	0	0
4	0	0	2/7	-3/7	1	0
20	0	0	1	0	0	1

$B = \{1, 2, 5, 6\}$
$x = [3\ 8\ 0\ 0\ 4\ 20]$
$w = -19$

The current solution is in C. Active constraints are (3) and (4).

Iteration 3.

19	0	0	-1/7	5/7	0	0
8	0	1	1/7	2/7	0	0
3	1	0	-3/7	1/7	0	0
4	0	0	**2/7**	-3/7	1	0
20	0	0	1	0	0	1

$B = \{1, 2, 5, 6\}$
$x = [3\ 8\ 0\ 0\ 4\ 20]$
$w = -19$

21	0	0	0	1/2	1/2	0
6	0	1	0	1/2	-1/2	0
9	1	0	0	-1/2	3/2	0
14	0	0	1	-3/2	7/2	0
6	0	0	0	3/2	-7/2	1

$B = \{1, 2, 3, 6\}$
$x = [9\ 6\ 14\ 0\ 0\ 6]$
$w = -21$

The current solution is in D. Active constraints are (4) and (5).

The simplex algorithm stops because the optimality conditions are satisfied. The optimal value is $z^* = -w^* = 21$.

Exercise 1.2: finite optimal solution (more than 2 dimensions).

Given the following LP,

$$\begin{aligned} \text{maximize } z = & 2x_1 + 3x_2 + 4x_3 + 5x_4 \\ \text{s.t. } & x_1 + x_2 - x_3 + x_4 \leq 10 \\ & x_1 + 2x_2 \leq 8 \\ & x_3 + x_4 \leq 20 \\ & x \geq 0 \end{aligned}$$

compute its optimal solution with the simplex algorithm.

Solution.

First of all, we rewrite the model in standard form (minimization form and $\leq$ inequalities).

$$\begin{aligned} \text{minimize } w = & -2x_1 - 3x_2 - 4x_3 - 5x_4 \\ \text{s.t. } & x_1 + x_2 - x_3 + x_4 + x_5 \leq 10 \\ & x_1 + 2x_2 + x_6 \leq 8 \\ & x_3 + x_4 + x_7 \leq 20 \\ & x \geq 0 \end{aligned}$$

The corresponding tableau is as follows.

0	-2	-3	-4	-5	0	0	0
10	1	1	-1	1	1	0	0
8	1	2	0	0	0	1	0
20	0	0	1	1	0	0	1

Column selection policy: minimum reduced cost.

Iteration 1. Selecting a column with minimum reduced cost, we pivot an the element indicated on the left, and we obtain the tableau on the right.

0	-2	-3	-4	-5	0	0	0
10	1	1	-1	**1**	1	0	0
8	1	2	0	0	0	1	0
20	0	0	1	1	0	0	1

$B = \{5, 6, 7\}$
$x = [0\ 0\ 0\ 0\ 10\ 8\ 20]$
$w = 0$

50	3	2	-9	0	5	0	0
10	1	1	-1	1	1	0	0
8	1	2	0	0	0	1	0
10	-1	-1	2	0	-1	0	1

$B = \{4, 6, 7\}$
$x = [0\ 0\ 0\ 10\ 0\ 8\ 10]$
$w = -50$

Iteration 2. With the same criterion we obtain the following tableaux.

50	3	2	-9	0	5	0	0
10	1	1	-1	1	1	0	0
8	1	2	0	0	0	1	0
10	-1	-1	**2**	0	-1	0	1

$B = \{4, 6, 7\}$
$x = [0\ 0\ 0\ 10\ 0\ 8\ 10]$
$w = -50$

95	-3/2	-5/2	0	0	1/2	0	9/2
15	1/2	1/2	0	1	1/2	0	1/2
8	1	2	0	0	0	1	0
5	-1/2	-1/2	1	0	-1/2	0	1/2

$B = \{3, 4, 6\}$
$x = [0\ 0\ 5\ 15\ 0\ 8\ 0]$
$w = -95$

Iteration 3.

95	-3/2	-5/2	0	0	1/2	0	9/2
15	1/2	1/2	0	1	1/2	0	1/2
8	1	**2**	0	0	0	1	0
5	-1/2	-1/2	1	0	-1/2	0	1/2

$B = \{3, 4, 6\}$
$x =$ [0 0 5 15 0 8 0]
$w = -95$

105	-1/4	0	0	0	1/2	5/4	9/2
13	1/4	0	0	1	1/2	-1/4	1/2
4	1/2	1	0	0	0	1/2	0
7	-1/4	0	1	0	-1/2	1/4	1/2

$B = \{2, 3, 4\}$
$x =$ [0 4 7 13 0 0 0]
$w = -105$

Iteration 4.

105	-1/4	0	0	0	1/2	5/4	9/2
13	1/4	0	0	1	1/2	-1/4	1/2
4	**1/2**	1	0	0	0	1/2	0
7	-1/4	0	1	0	-1/2	1/4	1/2

$B = \{2, 3, 4\}$
$x =$ [0 4 7 13 0 0 0]
$w = -105$

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

$B = \{1, 3, 4\}$
$x =$ [8 0 9 11 0 0 0]
$w = -107$

The optimal solution has been reached in four iterations.

Column selection policy: maximum improvement.

A different strategy can be used to select the entering variable. For instance, we can evaluate the improvement in the objective function value and we can select a column providing maximum benefit. The improvement in the objective value when pivoting on element on row i and column j is given by

$$\Delta z = \left| \frac{a_{i0} a_{0j}}{a_{ij}} \right| .$$

Iteration 1.

In our example, the starting tableau offers four possibilities (columns with $a_{0j} < 0$). The corresponding four pivot elements are shown in bold.

0	-2	-3	-4	-5	0	0	0
10	1	1	-1	**1**	1	0	0
8	**1**	**2**	0	0	0	1	0
20	0	0	**1**	1	0	0	1

The improvements in the objective function corresponding to each entering column in $\{1, 2, 3, 4\}$ are $\Delta z =$ [16 12 80 50]. Following the maximum improvement criterion, column 3 is selected.

0	-2	-3	-4	-5	0	0	0
10	1	1	-1	1	1	0	0
8	1	2	0	0	0	1	0
20	0	0	**1**	1	0	0	1

$B = \{5, 6, 7\}$
$x =$ [0 0 0 0 10 8 10]
$w = 0$

80	-2	-3	0	-1	0	0	4
30	1	1	0	2	1	0	1
8	1	2	0	0	0	1	0
20	0	0	1	1	0	0	1

$B = \{3, 5, 6\}$
$x =$ [0 0 20 0 30 8 0]
$w = -80$

Iteration 2.

At the next iteration we have three possibilities: columns 1, 2 and 4. They provide improvements equal to $\frac{8\times2}{1}$, $\frac{8\times3}{2}$ and $\frac{30\times1}{2}$, respectively. Therefore, column 1 is chosen.

80	-2	-3	0	-1	0	0	4
30	1	1	0	2	1	0	1
8	**1**	2	0	0	0	1	0
20	0	0	1	1	0	0	1

$B = \{3, 5, 6\}$
$x =$ [0 0 20 0 30 8 0]
$w = -80$

96	0	1	0	-1	0	2	4
22	0	-1	0	2	1	-1	1
8	1	2	0	0	0	1	0
20	0	0	1	1	0	0	1

$B = \{1, 3, 5\}$
$x =$ [8 0 20 0 22 0 0]
$w = -96$

Iteration 3.
At the next iteration we have only one possibility: column 4.

96	0	1	0	-1	0	2	4
22	0	-1	0	**2**	1	-1	1
8	1	2	0	0	0	1	0
20	0	0	1	1	0	0	1

$B = \{1, 3, 5\}$
$x =$ [8 0 20 0 22 0 0]
$w = -96$

107	0	1/2	0	0	1/2	3/2	9/2
11	0	-1/2	0	1	1/2	-1/2	1/2
8	1	2	0	0	0	1	0
9	0	1/2	1	0	-1/2	1/2	1/2

$B = \{1, 3, 4\}$
$x =$ [8 0 9 11 0 0 0]
$w = -107$

The algorithm stops: an optimal solution has been reached in three iterations.

Remark. In general, there is no guarantee about which rule allows the simplex algorithm to reach an optimal solution with a minimum number of iterations.

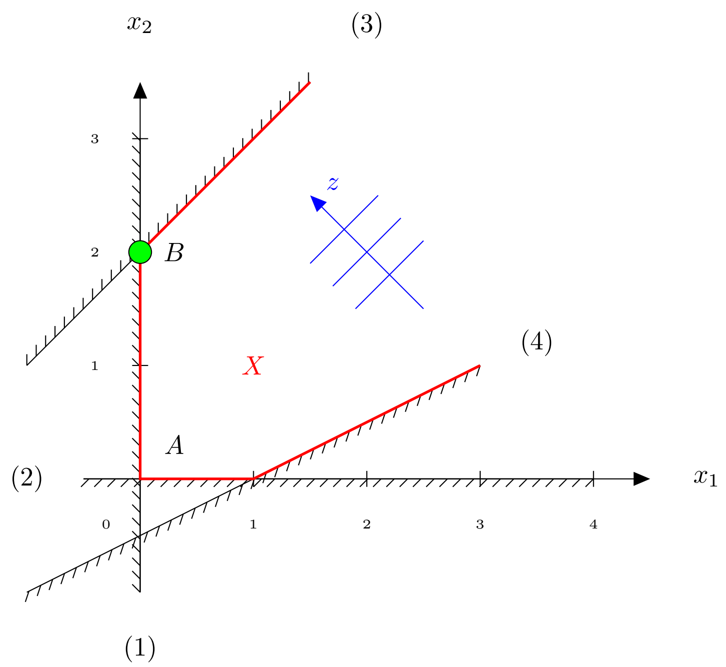
Exercise 1.3: unbounded polyhedron, finite optimal solution.

Given the following LP,

$$\begin{aligned} \text{minimize } z = & x_1 - x_2 \\ \text{s.t. } & x_1 - x_2 \geq -2 \\ & -x_1 + 2x_2 \geq -1 \\ & x \geq 0 \end{aligned}$$

compute its optimal solution by geometrical arguments and with the simplex algorithm.

Solution: geometrical construction.
By geometrical means we get the following solution.



The optimum is at point $B = [0, 2]$ with $z^* = -2$.

Solution with the simplex algorithm.
There is only one possible pivot, on column 2, row 1.

0	1	-1	0	0
2	-1	**1**	1	0
1	1	-2	0	1

$B = \{3, 4\}$
$x = [0\ 0\ 2\ 1]$
$z = 0$

2	0	0	1	0
2	-1	1	1	0
5	-1	0	2	1

$B = \{2, 4\}$
$x = [0\ 2\ 0\ 5]$
$z = -2$

The polyhedron is open, but not along the direction of optimization: hence a finite optimal solution exists.

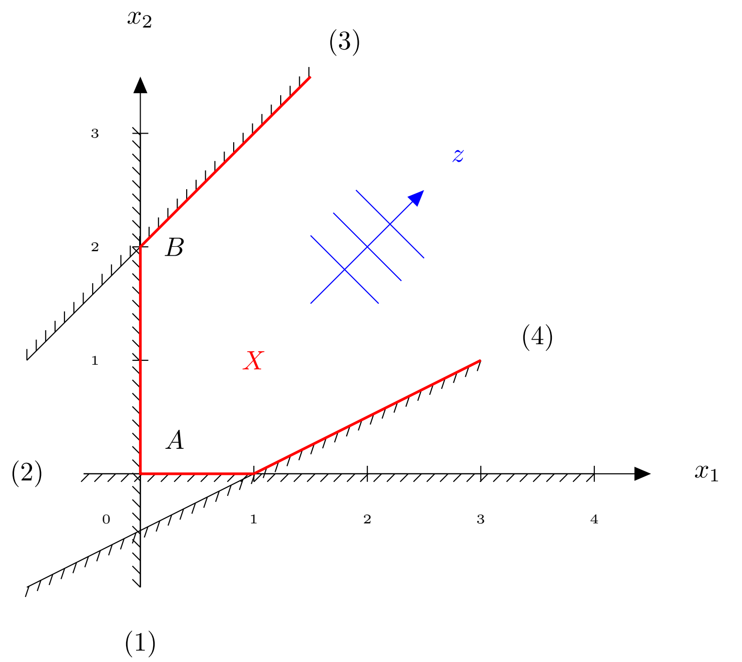
Exercise 1.4: unbounded problem (2 dimensions).

Given the LP,

$$\begin{aligned} \text{maximize } z =& x_1 + x_2 \\ \text{s.t. } & x_1 - x_2 \geq -2 \\ & -x_1 + 2x_2 \geq -1 \\ & x \geq 0 \end{aligned}$$

compute its optimal solution by geometrical arguments and with the simplex algorithm.

Solution: geometrical construction.
By geometrical means we get the following solution.



The polyhedron is open in the direction of z: the problem is unbounded.

Solution with the simplex algorithm.
With the simplex algorithm, after replacing the objective function z to be maximized with $w = -z$ to be minimized, we have two possibilities for pivoting, on column 1, row 2 and on column 2, row 1. Using - for instance - a lexicographical tie-break criterion, we select column 1.

0	-1	-1	0	0
2	-1	1	1	0
1	**1**	-2	0	1

$B = \{3,4\}$
$x = [0\ 0\ 2\ 1]$
$w = 0$

1	0	-3	0	1
3	0	-1	1	1
1	1	-2	0	1

$B = \{1,3\}$
$x = [1\ 0\ 3\ 0]$
$w = 1$

The tableau contains a column with negative reduced cost but with no positive entries. Then, the simplex algorithm detects that no finite optimal solution exists and stops.

Exercise 1.5: unbounded problem (more than 2 dimensions).

Given the following LP,

$$\begin{aligned} \text{maximize } z = & x_1 + x_2 + x_3 + x_4 \\ \text{s.t. } & x_1 + x_2 - x_3 - 2x_4 \leq 10 \\ & 2x_1 - 3x_2 + x_3 - x_4 \leq 8 \\ & x_1 - x_2 - x_3 + x_4 \leq 7 \\ & x \geq 0 \end{aligned}$$

solve it with the simplex algorithm.

Solution with the simplex algorithm.

First, we replace the objective function z to be maximized with $w = -z$ to be minimized.

Iteration 1.

We select, for instance, the column yielding maximum improvement in the objective function.

0	-1	-1	-1	-1	0	0	0
10	1	**1**	-1	-2	1	0	0
8	2	-3	1	-1	0	1	0
7	1	-1	-1	1	0	0	1

$B = \{5, 6, 7\}$
$x = [0\ 0\ 0\ 0\ 10\ 8\ 7]$
$w = 0$

10	0	0	-2	-3	-1	0	0
10	1	1	-1	-2	1	0	0
38	5	0	-2	-7	0	1	0
17	2	0	-2	-1	0	0	1

$B = \{2, 6, 7\}$
$x = [0\ 10\ 0\ 0\ 0\ 38\ 17]$
$w = -10$

The tableau contains two columns with negative reduced cost but with no positive entries: the problem is unbounded.

Exercise 1.6: degeneration (two dimensions).

Given the LP

$$
\begin{aligned}
\text{maximize } z = & x_1 + 2x_2 \\
\text{s.t. } & x_2 \le 2x_1 + 2 \\
& x_2 \le x_1 + 2 \\
& x_2 \le \frac{1}{2}x_1 + 2 \\
& x_1 \le 4 \\
& x \ge 0
\end{aligned}
$$

solve it with the simplex algorithm and by geometrical arguments.

Solution with the simplex algorithm.
To put the problem in standard form, we insert the slack variables and we replace the objective function z to be maximized with $w = -z$ to be minimized.

$$
\begin{aligned}
\text{minimize } w = & -x_1 - 2x_2 \\
\text{s.t. } & -2x_1 + x_2 + x_3 = 2 \\
& -x_1 + x_2 + x_4 = 2 \\
& -x_1 + 2x_2 + x_5 = 4 \\
& x_1 + x_6 = 4 \\
& x \ge 0
\end{aligned}
$$

Iteration 1. We select the column with minimum reduced cost. At the first iteration there are three possible equivalent choices for the pivot row. As a tie-break rule, we choose the row according to the constraint ordering. Therefore we select the pivot on row 1.

0	-1	-2	0	0	0	0
2	-2	**1**	1	0	0	0
2	-1	1	0	1	0	0
4	-1	2	0	0	1	0
4	1	0	0	0	0	1

$B = \{3, 4, 5, 6\}$
$x = [0\ 0\ 2\ 2\ 4\ 4]$
$w = 0$

4	-5	0	2	0	0	0
2	-2	1	1	0	0	0
0	1	0	-1	1	0	0
0	3	0	-2	0	1	0
4	1	0	0	0	0	1

$B = \{2, 4, 5, 6\}$
$x = [0\ 2\ 0\ 0\ 0\ 4]$
$w = -4$

We observe that two variables, namely x_4 and x_5, are null even if they are basic.
Iteration 2.

4	-5	0	2	0	0	0
2	-2	1	1	0	0	0
0	**1**	0	-1	1	0	0
0	3	0	-2	0	1	0
4	1	0	0	0	0	1

$B = \{2, 4, 5, 6\}$
$x = [0\ 2\ 0\ 0\ 0\ 4]$
$w = -4$

4	0	0	-3	5	0	0
2	0	1	-1	2	0	0
0	1	0	-1	1	0	0
0	0	0	1	-3	1	0
4	0	0	1	-1	0	1

$B = \{1, 2, 5, 6\}$
$x = [0\ 2\ 0\ 0\ 0\ 4]$
$w = -4$

In this iteration the base has changed but the solution has not.
Iteration 3.

4	0	0	-3	5	0	0
2	0	1	-1	2	0	0
0	1	0	-1	1	0	0
0	0	0	**1**	-3	1	0
4	0	0	1	-1	0	1

$B = \{1, 2, 5, 6\}$
$x = [0\ 2\ 0\ 0\ 0\ 4]$
$w = -4$

4	0	0	0	-4	3	0
2	0	1	0	-1	1	0
0	1	0	0	-2	1	0
0	0	0	1	-3	1	0
4	0	0	0	2	-1	1

$B = \{1, 2, 3, 6\}$
$x = [0\ 2\ 0\ 0\ 0\ 4]$
$w = -4$

Again, a different basis but the same solution.
Iteration 4.

4	0	0	0	-4	3	0
2	0	1	0	-1	1	0
0	1	0	0	-2	1	0
0	0	0	1	-3	1	0
4	0	0	0	**2**	-1	1

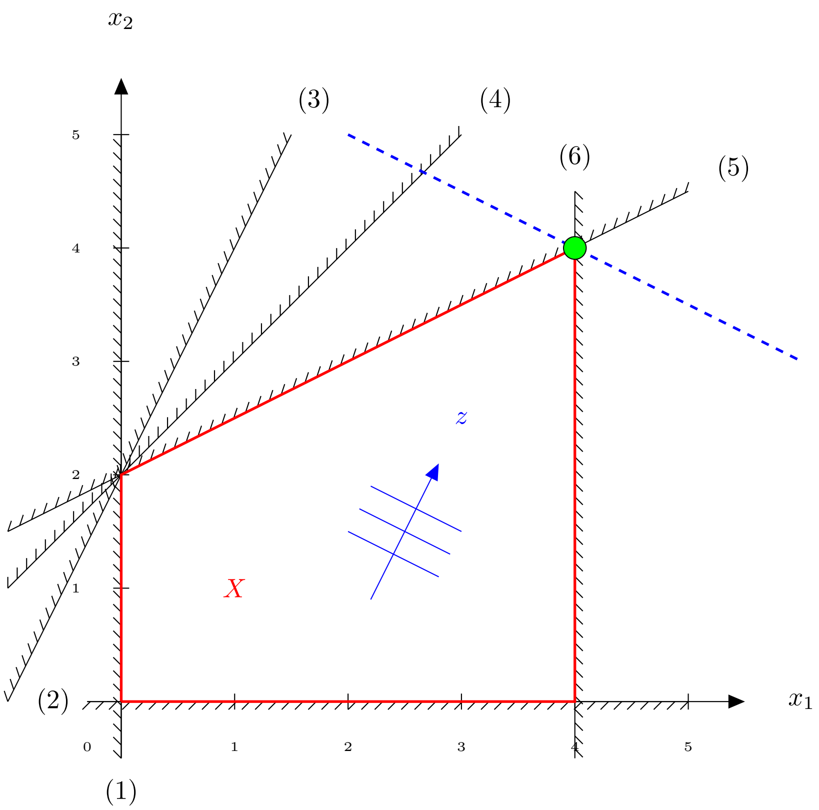
$B = \{1, 2, 3, 6\}$
$x = [0\ 2\ 0\ 0\ 0\ 4]$
$w = -4$

12	0	0	0	0	1	2
4	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$
4	1	0	0	0	0	1
6	0	0	1	0	$-\frac{1}{2}$	$\frac{3}{2}$
2	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$

$B = \{1, 2, 3, 4\}$
$x = [4\ 4\ 6\ 2\ 0\ 0]$
$w = -12$

Finally, we have reached the optimal solution.

Geometrical construction.
By geometrical means we get the following representation of the problem.



If the algorithm had started in the other direction at the beginning, selecting x_1 as the entering variable, the optimal solution would have been found in only two iterations, instead of four.

Exercise 1.7: degeneration (more than two dimensions).

Solve the following LP with the simplex algorithm.

$$\begin{aligned} \text{maximize } z = & 2x_1 + 3x_2 + 4x_3 + 5x_4 + 10 \\ \text{s.t. } & 2x_1 + x_2 - x_3 \le 8 \\ & 2x_2 + x_3 - x_4 \le 10 \\ & -x_1 + 2x_3 + x_4 \le 10 \\ & x_1 - x_2 + 2x_4 \le 12 \\ & x \ge 0 \end{aligned}$$

Solution.

Iteration 1. We select the column with minimum reduced cost.

10	-2	-3	-4	-5	0	0	0	0
8	2	1	-1	0	1	0	0	0
10	0	2	1	-1	0	1	0	0
10	-1	0	2	1	0	0	1	0
12	1	-1	0	**2**	0	0	0	1

$B = \{5, 6, 7, 8\}$
$x =$ [0 0 0 0 8 10 10 12]
$w = -10$

40	1/2	-11/2	-4	0	0	0	0	5/2
8	2	1	-1	0	1	0	0	0
16	1/2	3/2	1	0	0	1	0	1/2
4	-3/2	1/2	2	0	0	0	1	-1/2
6	1/2	-1/2	0	1	0	0	0	1/2

$B = \{4, 5, 6, 7\}$
$x =$ [0 0 0 6 8 16 4 0]
$w = -40$

Iteration 2. We select column 2, with minimum reduced cost. Row 1 and 3 can be selected indifferently. We select row 3.

40	1/2	-11/2	-4	0	0	0	0	5/2
8	2	1	-1	0	1	0	0	0
16	1/2	3/2	1	0	0	1	0	1/2
4	-3/2	**1/2**	2	0	0	0	1	-1/2
6	1/2	-1/2	0	1	0	0	0	1/2

$B = \{4, 5, 6, 7\}$
$x =$ [0 0 0 6 8 16 4 0]
$w = -40$

84	-16	0	18	0	0	0	11	-3
0	5	0	-5	0	1	0	-2	1
4	5	0	-5	0	0	1	-3	2
8	-3	1	4	0	0	0	2	-1
10	-1	0	2	1	0	0	1	0

$B = \{2, 4, 5, 6\}$
$x =$ [0 8 0 10 0 4 0 0]
$w = -84$

The variable x_5 is null even though it is basic; the solution is degenerate.

Iteration 3.

84	-16	0	18	0	0	0	11	-3
0	**5**	0	-5	0	1	0	-2	1
4	5	0	-5	0	0	1	-3	2
8	-3	1	4	0	0	0	2	-1
10	-1	0	2	1	0	0	1	0

$B = \{2, 4, 5, 6\}$
$x =$ [0 8 0 10 0 4 0 0]
$w = -84$

84	0	0	2	0	16/5	0	23/5	1/5
0	1	0	-1	0	1/5	0	-2/5	1/5
4	0	0	0	0	-1	1	-1	1
8	0	1	1	0	3/5	0	4/5	-2/5
10	0	0	1	1	1/5	0	3/5	1/5

$B = \{1, 2, 4, 6\}$
$x =$ [0 8 0 10 0 4 0 0]
$w = -84$

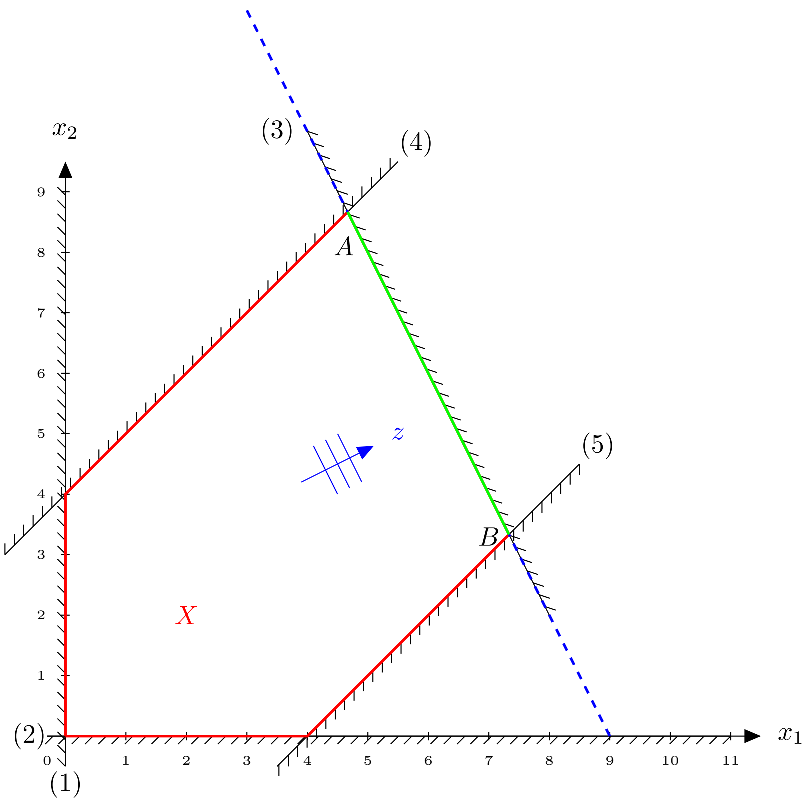
The optimal solution is the same as the previous one. But now the optimality conditions are satisfied; with the previous basis they were not, although the solution was already optimal.

Exercise 1.8: multiple optima.

Solve the following LP problem geometrically and with the simplex algorithm.

$$
\begin{aligned}
\text{maximize } z =& 2x_1 + x_2 \\
\text{s.t. } & 2x_1 + x_2 \leq 18 \\
& -x_1 + x_2 \leq 4 \\
& x_1 - x_2 \leq 4 \\
& x \geq 0
\end{aligned}
$$

Geometrical solution.
The geometrical representation of the problem is the following.



All points along the segment between A and B are equivalent and optimal. The endpoints A and B are the basic optimal solutions.

Solution with the simplex algorithm.
Iteration 1. We select the column with minimum reduced cost.

0	-2	-1	0	0	0
18	2	1	1	0	0
4	-1	1	0	1	0
4	**1**	-1	0	0	1

$B = \{3, 4, 5\}$
$x = [0\ 0\ 18\ 4\ 4]$
$w = 0$

8	0	-3	0	0	2
10	0	3	1	0	-2
8	0	0	0	1	1
4	1	-1	0	0	1

$B = \{1, 3, 4\}$
$x = [4\ 0\ 10\ 8\ 0]$
$w = -8$

Iteration 2.

8	0	-3	0	0	2
10	0	**3**	1	0	-2
8	0	0	0	1	1
4	1	-1	0	0	1

$B = \{1, 3, 4\}$
$x = [4\ 0\ 10\ 8\ 0]$
$w = -8$

18	0	0	1	0	0
10/3	0	1	1/3	0	-2/3
8	0	0	0	1	1
22/3	1	0	1/3	0	1/3

$B = \{1, 2, 4\}$
$x = [\frac{22}{3}\ \frac{10}{3}\ 0\ 8\ 0]$
$w = -18$

This solution corresponds to point B and it is optimal. However, variable x_5, which is non-basic, has null reduced cost. Hence, selecting it as the entering variable for a further iteration of the algorithm, leads to an equivalent basic solution, that is point A.

Iteration 3.

18	0	0	1	0	0
10/3	0	1	1/3	0	-2/3
8	0	0	0	1	**1**
22/3	1	0	1/3	0	1/3

$B = \{1, 2, 4\}$
$x = [\frac{22}{3}\ \frac{10}{3}\ 0\ 8\ 0]$
$w = -18$

18	0	0	1	0	0
26/3	0	1	1/3	2/3	0
8	0	0	0	1	1
14/3	1	0	1/3	-1/3	0

$B = \{1, 2, 5\}$
$x = [\frac{14}{3}\ \frac{26}{3}\ 0\ 0\ 8]$
$w = -18$

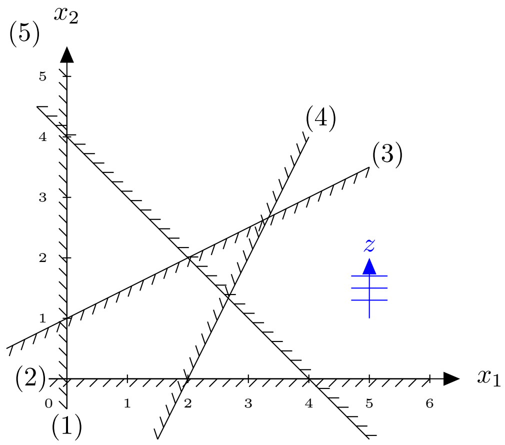
Exercise 1.9: infeasible starting basis, infeasible problem.

Solve the following LP geometrically and with the simplex algorithm.

$$
\begin{aligned}
\text{maximize } z =& x_2 \\
\text{s.t. } & x_1 - 2x_2 \leq -2 \\
& -2x_1 + x_2 \leq -4 \\
& x_1 + x_2 \leq 4 \\
& x \geq 0
\end{aligned}
$$

Geometrical solution.

The geometrical representation of the problem is the following.



The polyhedron defined by the constraints is empty: no feasible solution exists.

Solution with the simplex algorithm: artificial variables.

We rewrite the problem so that all right-hand-sides coefficients are non-negative. After inserting slack or surplus variables, we obtain the following LP.

$$
\begin{aligned}
\text{maximize } z =& x_2 \\
\text{s.t. } & -x_1 + 2x_2 - x_3 = 2 \\
& 2x_1 - x_2 - x_4 = 4 \\
& x_1 + x_2 + x_5 = 4 \\
& x \geq 0
\end{aligned}
$$

Now, we insert artificial variables in constraints with index (3) and (4). This is not needed in constraint (5) because the column of x_5 already satisfies the canonical form conditions. The objective function of the artificial problem asks for the minimization of the sum of the artificial variables.

$$
\begin{aligned}
\text{minimize } w =& u_6 + u_7 \\
\text{s.t. } & -x_1 + 2x_2 - x_3 + u_6 = 2 \\
& 2x_1 - x_2 - x_4 + u_7 = 4 \\
& x_1 + x_2 + x_5 = 4 \\
& x, u \geq 0
\end{aligned}
$$

The artificial problem is in standard form, but not in canonical form. The columns of u_6, u_7 and x_5 satisfy the first condition of canonical forms, but not the second. To enforce the secodn condition we have to replace u_6 and u_7 in the objective function. From the constraints we obtain

$$\begin{cases} u_6 = x_1 - 2x_2 + x_3 + 2 \\ u_7 = -2x_1 + x_2 + x_4 + 4 \end{cases}$$

and therefore

$$w = u_6 + u_7 = -x_1 - x_2 + x_3 + x_4 + 6.$$

The artificial problem is:

$$\begin{aligned} \text{minimize } w = &- x_1 - x_2 + x_3 + x_4 + 6 \\ \text{s.t. } &- x_1 + 2x_2 - x_3 + u_6 = 2 \\ &2x_1 - x_2 - x_4 + u_7 = 4 \\ &x_1 + x_2 + x_5 = 4 \\ &x, u \geq 0 \end{aligned}$$

The problem is now in strong canonical form with $u_6 = 2$, $u_7 = 4$ and $x_5 = 4$ as basic variables.

-6	-1	-1	1	1	0	0	0
2	-1	2	-1	0	0	1	0
4	**2**	-1	0	-1	0	0	1
4	1	1	0	0	1	0	0

$$B = \{5, 6, 7\} \;\; x = [0\,0\,0\,0\,4] \;\; u = [2\;4] \;\; w = 6$$

After pivoting on column 1, row 2, the following tableau is obtained.

-4	0	-3/2	1	1/2	0	0	1/2
4	0	3/2	-1	-1/2	0	1	1/2
2	1	-1/2	0	-1/2	0	0	1/2
2	0	**3/2**	0	1/2	1	0	-1/2

$$B = \{1, 5, 6\} \;\; x = [2\,0\,0\,0\,2] \;\; u = [4\;0] \;\; w = 4$$

The artificial variable u_7 is now non-basic. After pivoting on column 2, row 3, the following tableau is obtained.

-2	0	0	1	1	1	0	0
2	0	0	-1	-1	-1	1	1
8/3	1	0	0	-1/3	1/3	0	1/3
4/3	0	1	0	1/3	2/3	0	-1/3

$$B = \{1, 2, 6\} \;\; x = [\tfrac{8}{3}\;\tfrac{4}{3}\;0\,0\,0] \;\; u = [2\;0] \;\; w = 2$$

Optimality conditions are satisfied: the artificial objective cannot be further reduced. Since $w^* = 2 > 0$, the original problem is infeasible.

Solution with the simplex algorithm: Balinsky-Gomory method.

After rewriting the LP model in standard form, we observe that the starting solution is infeasible.

0	0	-1	0	0	0
-2	1	-2	1	0	0
-4	-2	1	0	1	0
4	1	1	0	0	1

$$B = \{3, 4, 5\} \;\; x = [0\,0\,-2\,-4\,4] \;\; w = 0$$

Two basic variables have a negative value. We define an auxiliary problem, by temporary using one of the violated constraints as an objective. We select the second constraint, because it is violated by the largest amount.

-4	-2	1	0	1	0
-2	1	-2	1	0	0
4	1	1	0	0	1
0	0	-1	0	0	0

Iteration 1.

We run the simplex algorithm on this auxiliary problem. Column 1 enters the basis. We avoid pivoting on the objective function row (we would lose the canonical form), on violated constraints (we will consider them at later iterations) and on negative entries (we would lose the feasibility on their rows). Therefore, we are left with a unique possibility: pivoting on the last constraint.

-4	-2	1	0	1	0
-2	1	-2	1	0	0
4	**1**	1	0	0	1
0	0	-1	0	0	0

$B = \{3, 4, 5\}$
$x = [0\ 0\ 18\ 4\ 4]$
$w = 0$

4	0	3	0	1	2
-6	0	-3	1	0	-1
4	1	1	0	0	1
0	0	-1	0	0	0

$B = \{1, 3, 4\}$
$x = [4\ 0\ -6\ 4\ 0]$
$w = 0$

Now the constraint we are using as an auxiliary objective is satisfied (the value in the top-left corner of the tableau is now positive). So, we can restore the original tableau.

0	0	-1	0	0	0
-6	0	-3	1	0	-1
4	0	3	0	1	2
4	1	1	0	0	1

$$B = \{1, 3, 4\} \quad x = [4\ 0\ -6\ 4\ 0] \quad w = 0$$

The current solution is still infeasible, because there is another violated constraint. We use it as a temporary objective function and we define a new auxiliary problem.

-6	0	-3	1	0	-1
4	0	3	0	1	2
4	1	1	0	0	1
0	0	-1	0	0	0

$$B = \{1, 3, 4\} \quad x = [4\ 0\ -6\ 4\ 0] \quad w = 0$$

Iteration 2.

Column 2 enters the basis.

-6	0	-3	1	0	-1
4	0	**3**	0	1	2
4	1	1	0	0	1
0	0	-1	0	0	0

$B = \{1, 3, 4\}$
$x = [4\ 0\ -6\ 4\ 0]$
$w = 0$

-2	0	0	1	1	1
4/3	0	1	0	1/3	2/3
8/3	1	0	0	-1/3	1/3
4/3	0	0	0	1/3	2/3

$B = \{1, 2, 3\}$
$x = [\frac{8}{3}\ \frac{4}{3}\ -2\ 0\ 0]$
$w = -\frac{4}{3}$

The auxiliary problem cannot be optimized further, because the optimality conditions are satisfied: all reduced costs are non-negative. Since the constraint is still violated, the problem is infeasible. The top row of the tableau corresponds to the equation

$$x_3 + x_4 + x_5 = -2$$

that cannot be satisfied without violating the non-negativity conditions on the variables.

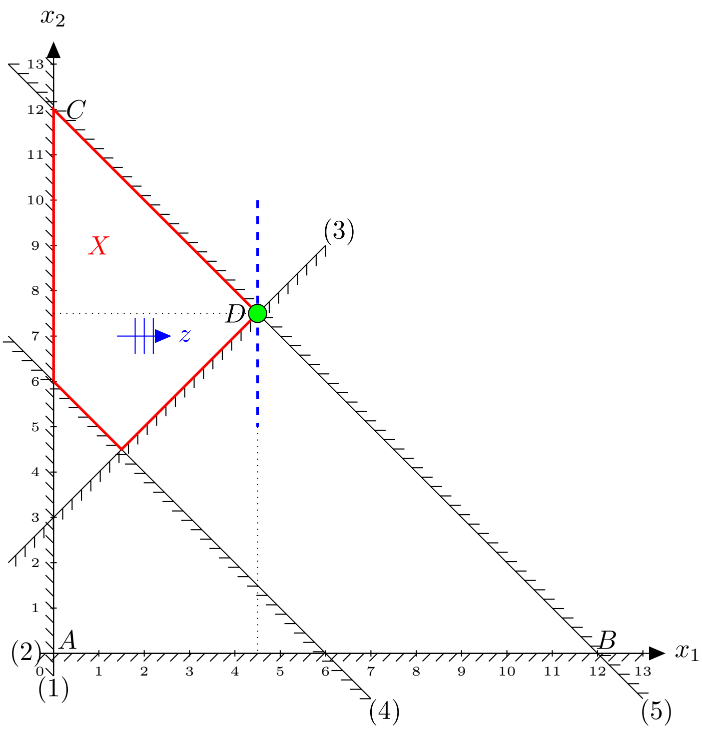
Exercise 1.10: infeasible starting basis, feasible problem.

Solve the following LP geometrically and with the simplex algorithm.

$$
\begin{aligned}
\text{maximize } z =& x_1 \\
\text{s.t. } & x_1 - x_2 \leq -3 \\
& -x_1 - x_2 \leq -6 \\
& x_1 + x_2 \leq 12 \\
& x \geq 0
\end{aligned}
$$

Geometrical solution.

The geometrical representation of the problem is the following.



The origin is not feasible. The optimal solution is at $(\frac{9}{2}, \frac{15}{2})$.

Solution with the simplex algorithm: artificial variables.

We rewrite the model so that all right-hand-side coefficients are non-negative.

$$
\begin{aligned}
\text{maximize } z =& x_1 \\
\text{s.t. } & -x_1 + x_2 \geq 3 \\
& x_1 + x_2 \geq 6 \\
& x_1 + x_2 \leq 12 \\
& x \geq 0
\end{aligned}
$$

We rewrite the model so that all right-hand-side coefficients are non-negative. The we write the problem in standard form.

$$
\begin{aligned}
\text{minimize } z' = & -x_1 \\
\text{s.t. } & -x_1 + x_2 - x_3 = 3 \\
& x_1 + x_2 - x_4 = 6 \\
& x_1 + x_2 + x_5 = 12 \\
& x \geq 0
\end{aligned}
$$

Inserting artificial variables u_6 and u_7 in constraints (3) and (4), we obtain the following artificial problem:

$$
\begin{aligned}
\text{minimize } w = & 9 - 2x_2 + x_3 + x_4 \\
\text{s.t. } & -x_1 + x_2 - x_3 + u_6 = 3 \\
& x_1 + x_2 - x_4 + u_7 = 6 \\
& x_1 + x_2 + x_5 = 12 \\
& x, u \geq 0
\end{aligned}
$$

The problem is in strong canonical form with basic variables x_5, u_6 and u_7.

-9	0	-2	1	1	0	0	0
3	-1	**1**	-1	0	0	1	0
6	1	1	0	-1	0	0	1
12	1	1	0	0	1	0	0

$$B = \{5, 6, 7\} \;\; x = [0\;0\;0\;0\;12] \;\; u = [3\;6] \;\; w = 9$$

After pivoting on column 2, row 1 the following tableau is obtained:

-3	-2	0	-1	1	0	2	0
3	-1	1	-1	0	0	1	0
3	**2**	0	1	-1	0	-1	1
9	1	0	1	0	1	-1	0

$$B = \{2, 5, 7\} \;\; x = [0\;3\;0\;0\;9] \;\; u = [0\;3] \;\; w = 3$$

After pivoting on column 1, row 2 the following tableau is obtained:

0	0	0	0	0	0	1	1
9/2	0	1	-1/2	-1/2	0	1/2	1/2
3/2	1	0	1/2	-1/2	0	-1/2	1/2
6	0	0	0	1	1	0	-1

$$B = \{1, 2, 5\} \;\; x = [\tfrac{3}{2}\;\tfrac{9}{2}\;0\;0\;6] \;\; u = [0\;0] \;\; w = 0$$

Now, to write the original problem in canonical form, it is necessary to replace basic variables with linear combinations of non-baisc variables in the expression of z. From constraint on row 2,

$$x_1 = -\frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{3}{2}$$

and therefore

$$z = x_1 = -\frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{3}{2}.$$

The second phase of the simplex algorithm can start from the following tableau, where all feasibility conditions are satisfied.

3/2	0	0	1/2	-1/2	0
9/2	0	1	-1/2	-1/2	0
3/2	1	0	1/2	-1/2	0
6	0	0	0	**1**	1

$$B = \{1,2,5\} \;\; x = [\tfrac{3}{2}\; \tfrac{9}{2}\; 0\; 0\; 6] \;\; z = \tfrac{3}{2}$$

After pivoting on column 4, row 3, the following tableau is obtained.

9/2	0	0	1/2	0	1/2
15/2	0	1	-1/2	0	1/2
9/2	1	0	1/2	0	1/2
6	0	0	0	1	1

$$B^* = \{1,2,4\} \;\; x^* = [\tfrac{9}{2}\; \tfrac{15}{2}\; 0\; 6\; 0] \;\; z^* = \tfrac{9}{2}$$

Optimality conditions are satisfied. The solution is feasible and optimal.

Solution with the simplex algorithm: Balinsky-Gomory method.

After rewriting the LP model in standard form, we observe that the starting solution (point A) is infeasible.

0	-1	0	0	0	0
-3	1	-1	1	0	0
-6	-1	-1	0	1	0
12	1	1	0	0	1

$$B = \{3,4,5\} \;\; x = [0\; 0\; -3\; -6\; 12] \;\; w = 0$$

Two basic variables have a negative value. We define an auxiliary problem, by temporary using one of the violated constraints as an objective. We select the second constraint, because it is violated by the largest amount.

-6	-1	-1	0	1	0
-3	1	-1	1	0	0
12	1	1	0	0	1
0	-1	0	0	0	0

Iteration 1.

We run the simplex algorithm on this auxiliary problem. We can select either column 1 or column 2 to enter the basis. Assume we select column 1 (you can try to solve the exercise by starting the other way).

We avoid pivoting on the objective function row (we would lose the canonical form), on violated constraints like constraint 1 (we will consider them at later iterations). Therefore we must pivot on the third constraint.

-6	-1	-1	0	1	0
-3	1	-1	1	0	0
12	**1**	1	0	0	1
0	-1	0	0	0	0

$B = \{3,4,5\}$
$x = [0\; 0\; -3\; -6\; 12]$
$w = 0$

6	0	0	0	1	1
-15	0	-2	1	0	-1
12	1	1	0	0	1
12	0	1	0	0	1

$B = \{1,3,4\}$
$x = [12\; 0\; -15\; 6\; 0]$
$w = -12$

Now the constraint is satisfied (the value in the top-left corner of the tableau is now positive). So, we can restore the original tableau.

12	0	1	0	0	1
-15	0	-2	1	0	-1
6	0	0	0	1	1
12	1	1	0	0	1

$$B = \{1, 3, 4\} \;\; x = [12\ 0\ {-15}\ 6\ 0] \;\; w = -12$$

The current solution (point B) is still infeasible, because there is another violated constraint. We use it as a temporary objective function and we define a new auxiliary problem.

-15	0	-2	1	0	-1
6	0	0	0	1	1
12	1	1	0	0	1
12	0	1	0	0	1

$$B = \{1, 3, 4\} \;\; x = [12\ 0\ {-15}\ 6\ 0] \;\; w = -12$$

Iteration 2.

Column 2 must enter the basis.

-15	0	-2	1	0	-1
6	0	0	0	1	1
12	1	**1**	0	0	1
12	0	1	0	0	1

$B = \{1, 3, 4\}$
$x = [12\ 0\ {-15}\ 6\ 0]$
$w = -12$

9	2	0	1	0	1
6	0	0	0	1	1
12	1	1	0	0	1
0	-1	0	0	0	0

$B = \{2, 3, 4\}$
$x = [0\ 12\ 9\ 6\ 0]$
$w = 0$

The constraint is no longer violated and we can restore the tableau of the original problem.

0	-1	0	0	0	0
9	2	0	1	0	1
6	0	0	0	1	1
12	1	1	0	0	1

$$B = \{2, 3, 4\} \;\; x = [0\ 12\ 9\ 6\ 0] \;\; w = 0$$

The search for feasibility has succeeded (point C). From now on, the simplex algorithm proceeds with the search for optimality.

Iteration 3.

Column 1 must enter the basis.

0	-1	0	0	0	0
9	**2**	0	1	0	1
6	0	0	0	1	1
12	1	1	0	0	1

$B = \{2, 3, 4\}$
$x = [0\ 12\ 9\ 6\ 0]$
$w = 0$

9/2	0	0	1/2	0	1/2
9/2	1	0	1/2	0	1/2
6	0	0	0	1	1
15/2	0	1	-1/2	0	1/2

$B = \{1, 2, 4\}$
$x = [\frac{9}{2}\ \frac{15}{2}\ 0\ 6\ 0]$
$w = -\frac{9}{2}$

The optimal value is $z^* = -w^* = \frac{9}{2}$ (point D).

Exercise 1.11: infeasible starting basis, degenerate artificial problem. Exercise taken from Caramia et al, Ricerca operativa, Ed. isedi.

Initialize the following LP with the artificial variables method.

$$\begin{aligned} \text{minimize } z = & 6x_1 + 2x_2 - 8x_3 + 6x_4 \\ \text{s.t. } & 2x_1 - 7x_2 - 3x_3 + 4x_4 = 6 \\ & 5x_1 + 3x_2 + 6x_3 + 8x_4 = 12 \\ & x \geq 0 \end{aligned}$$

The artificial problem is the following.

$$\begin{aligned} \text{minimize } w = & 18 - 7x_1 + 4x_2 - 3x_3 - 12x_4 \\ \text{s.t. } & 2x_1 - 7x_2 - 3x_3 + 4x_4 + u_5 = 6 \\ & 5x_1 + 3x_2 + 6x_3 + 8x_4 + u_6 = 12 \\ & x \geq 0 \end{aligned}$$

-18	-7	4	-3	-12	0	0
6	2	-7	-3	4	1	0
12	5	3	6	**8**	0	1

$$B = \{5, 6\} \;\; x = [0\,0\,0\,0] \;\; u = [6\;12] \;\; w = 18$$

After pivoting on column 4, row 2, the following tableau is obtained.

0	1/2	17/2	6	0	0	3/2
0	**-1/2**	-17/2	-6	0	1	-1/2
3/2	5/8	3/8	3/4	1	0	1/8

$$B = \{4, 5\} \;\; x = [0\,0\,0\,\tfrac{3}{2}] \;\; u = [0\;0] \;\; w = 0$$

The infeasibility has been repaired ($w = 0$) but the optimal basis of the artificial problem still contains one of the artificial variables (u_5). To force u_5 to leave the basis another pivot step is needed. It can be done on any column corresponding to an original non-basic variable x_1, x_2 or x_3 and on row 1, where u_5 is basic. After pivoting on column 1, row 1, the following tableau is obtained.

0	0	0	0	0	1	1
0	1	17	12	0	-2	1
3/2	0	-41/4	-27/4	1	5/4	-1/2

$$B = \{1, 4\} \;\; x = [0\,0\,0\,\tfrac{3}{2}] \;\; u = [0\;0] \;\; w = 0$$

Now a strong canonical form of the original problem is obtained.

$$\begin{aligned} \text{minimize } z = & 6x_1 + 2x_2 - 8x_3 + 6x_4 \\ \text{s.t. } & x_1 + 17x_2 + 12x_3 = 0 \\ & -\frac{41}{4}x_2 - \frac{27}{4}x_3 + x_4 = \frac{3}{2} \\ & x \geq 0 \end{aligned}$$

Since

$$\begin{cases} x_1 = -17x_2 - 12x_3 \\ x_4 = \frac{41}{4}x_2 + \frac{27}{4}x_3 + \frac{3}{2} \end{cases}$$

the objective function can be rewritten as

$$z = 6(-17x_2 - 12x_3) + 2x_2 - 8x_3 + 6(\frac{41}{4}x_2 + \frac{27}{4}x_3 + \frac{3}{2}) = 9 - \frac{77}{2}x_2 - \frac{79}{2}x_3$$

The corresponding tableau is

-9	0	-77/2	-79/2	0
0	1	17	12	0
3/2	0	-41/4	-27/4	1

$$B = \{1, 4\} \;\; x = [0\;0\;0\;\tfrac{3}{2}] \;\; z = 9$$

The initialization is over. The second phase of the simplex algorithm can start.