An improved primal cycle canceling algorithm for the assignment problem

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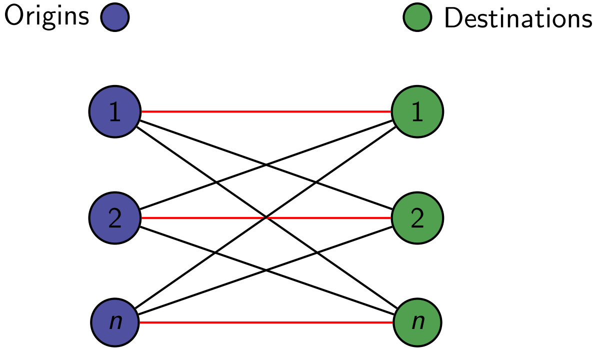
Outline

Introduction

- We consider the well-known assignment problem (AP) where in a weighted complete bipartite graph with n origins and n destinations, we search for a matching of size n, so that the sum of the arcs weights is minimum.



- We denote by c_{ij} the cost of the arc from origin i to destination j.

Introduction

- By introducing x_{ij} binary variables indicating whether an arc (i, j) is used or not - with i being an origin and j being a destination, a linear programming formulation of AP reads as follows

$$\min \sum_{i=1,\ldots,n} \sum_{j=1,\ldots,n} c_{ij} x_{ij} \tag{1}$$

$$\sum_{j=1,\ldots,n} x_{ij} = 1 \qquad \forall\, i = 1, \ldots, n \tag{2}$$

$$\sum_{i=1,\ldots,n} x_{ij} = 1 \qquad \forall\, j = 1, \ldots, n \tag{3}$$

$$x_{ij} \in \{0, 1\} \qquad \forall\, i, j = 1, \ldots, n \tag{4}$$

Due to the total unimodularity of the constraints matrix, constraints (4) can be substituted by

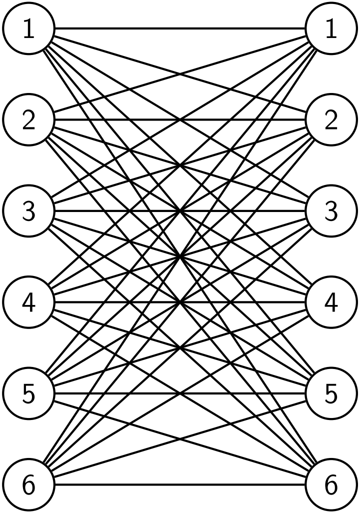
$$x_{ij} \geq 0 \qquad \forall\, i, j = 1, \ldots, n \tag{5}$$

Literature

- AP has been intensively studied in the XX century. We mention here among the proposed approaches **HM, the Hungarian method** (Kuhn-1955), **AA, the Auction algorithm** (Bertsekas-1988), **CS, the Cost Scaling algorithm** (Goldberg and Kennedy-1995), and the reference book by Burkard, Dell'Amico and Martello (2012).
- As AP is a special case of the transportation problem (TP), we cite also our recent **IIO, Iterated Inside Out algorithm** (2025) for TP.
- Top performances on AP are reached by the algorithms by Goldberg-Kennedy and Jonker-Volgenant (best implementation of HM), namely a **dual algorithm and a primal-dual algorithm**.
- Less efficient primal algorithms exist, e.g. Balinski and Gomory (1964) and Klein (1967 - **introducing the cycle canceling technique**) . Here we will **focus on a primal approach with cycle canceling**.

Primal approach with cycle canceling for AP

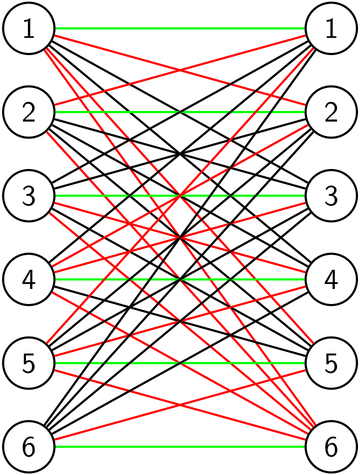
- Consider the following running example with $n = 6$ and the corresponding cost coefficients matrix



C_{ij} matrix					
9	11	14	11	7	5
6	15	13	13	10	9
12	13	6	8	8	7
11	9	10	12	10	6
7	12	14	10	9	6
9	14	13	11	8	7

Primal approach with cycle canceling for AP

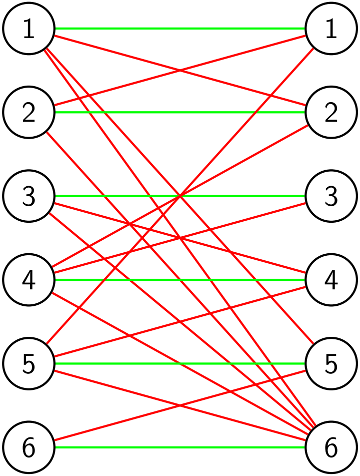
- Suppose an initial starting solution (1-1, 2-2, 3-3, 4-4, 5-5, 6-6 - colored in green) is given. Also, let color in red a suitable subset (shortlist) of the other arcs (e.g., min and second min cost arcs for each row and column).



C_{ij} matrix					
9	11	14	11	7	5
6	15	13	13	10	9
12	13	6	8	8	7
11	9	10	12	10	6
7	12	14	10	9	6
9	14	13	11	8	7

Primal approach with cycle canceling for AP

- To improve the solution, let us consider first red arcs only and the updated matrix C^*_{ij} with cost coefficients $c^*_{ij} = c_{ij} - c_{ii}\ \forall i, j$.

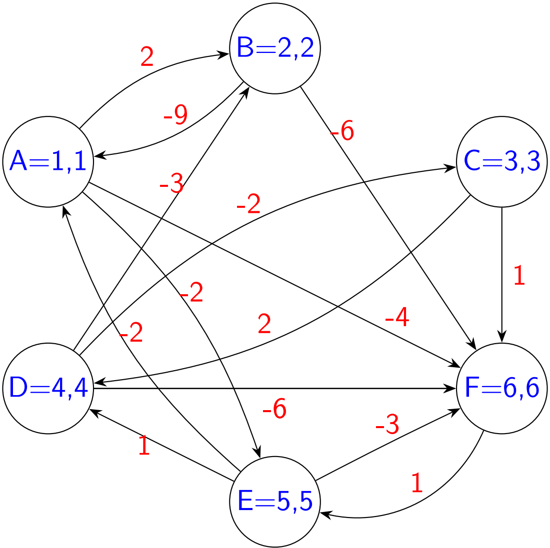


C_{ij} matrix					
9	11	.	.	7	5
6	15				9
		6	8		7
	9	10	12		6
7			10	9	6
				8	7

C^*_{ij} matrix					
0	2			-2	-4
-9	0				-6
		0	2		1
	-3	-2	0		-6
-2			1	0	-3
				1	0

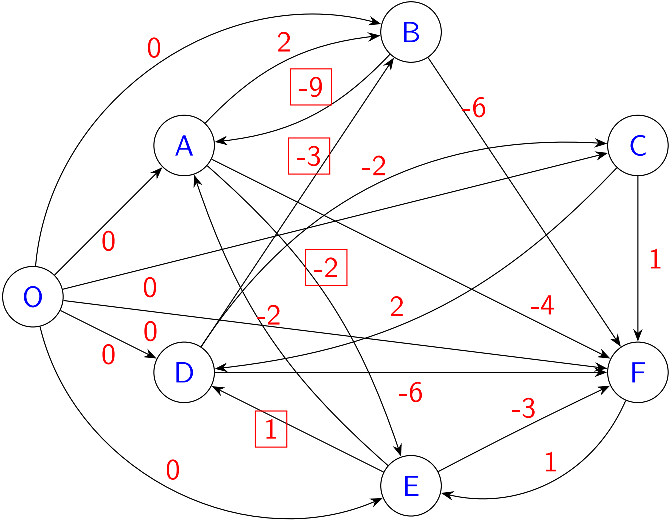
Primal approach with cycle canceling for AP

Finding an improvement to the previous matching corresponds to find a negative length cycle in this graph with vertices A,B,C,D,E;F.



Primal approach with cycle canceling for AP

It is sufficient to add a dummy vertex O and dummy arcs OA, OB, OC, OD, OE, OF (with null c^*_{ij}) and apply Bellman's Shortest Path algorithm.

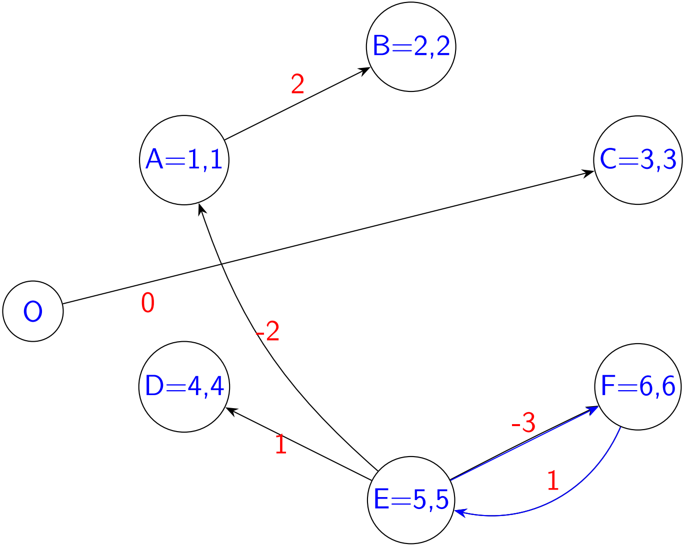


Proposed approach: the example

- Given the dummy vertex, the initial path length $\pi(i)$ for all vertices i $= \{A, B, C, D, E, F\}$ is $\pi(i) = 0$ and for all vertices $i = \{A, B, C, D, E, F\}$ the initial predecessor $P(i)$ is the dummy vertex, namely $P(i) = O$.
- Suppose we start the first iteration of Bellman's algorithm with vertex B and that the vertices are scanned following the order $B - A - C - D - E - F$.
- B: $\pi(A) = -9, \pi(F) = -6$, $P(A) = P(F) = B$.
  A: $\pi(B) = -7, \pi(E) = -14, \pi(F) = -13$, $P(B){=}P(E){=}P(F) = A$.
  C: no path update.
  D: no path update.
  E: $\pi(A) = -16, \pi(D) = -13, \pi(F) = -17$, $P(A){=}P(D){=}P(F) = E$.
  F: $\pi(E) = -20$, $P(E) = F$.
- At the end of the iteration, the predecessors are $P(A) = E$, $P(B) = A$, $P(C) = 0$, $P(D) = E$, $P(E) = F$, $P(F) = E$. Also, there is a negative cycle $E - F - E$.

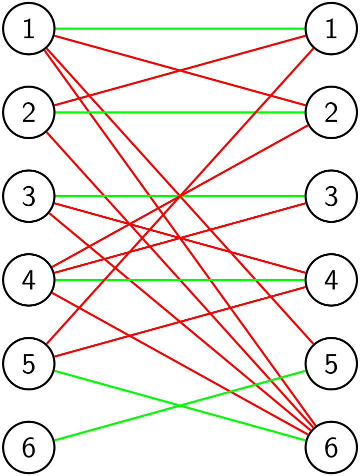
Primal approach with cycle canceling for AP

We can actually draw an auxiliary graph containing only the arcs induced by the predecessors where the detected cycle is colored in blue.



Primal approach with cycle canceling for AP

- Once a cycle (more cycles) is (are) detected, the matching is updated and the algorithm proceeds with the following iteration with the new matching below and relevant C_{ij} matrix



C_{ij} matrix					
9	11			7	5
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	9	10	12		6
7			10		6
				8	7

Primal approach with cycle canceling for AP

- For a given arcset and an initial matching, an optimal assignment can be obtained by means of the following procedure:
  1. Given the matching, compute for each arc (i, j) the updated weight c^*_{ij} and find at least one negative cycle by means of Bellman's algorithm on the relevant graph.
  2. IF a negative cycle is found, update the matching and GO TO 1 ELSE no negative cycle exists, the matching is optimal and the procedure stops.

- Rather than considering the complete bipartite arcset from the beginning, it makes sense to start at the beginning with a minimum cardinality subset (shortlist) of the arcs and find the optimal assignment for that shortlist.

- Then, progressively increase the size of the shortlist finding the related optimal assignment until the complete bipartite arcset is taken into account.

Primal approach with cycle canceling for AP

- A viable approach starts with a matching and an initial shortlist S_k with $k = 2$ composed by
  – the min k cost arcs outgoing from each origin and
  – the min k cost arcs reaching each destination.

- REMARK:
  At any time the following optimality test can be performed:
  Given shortlist S_k, the related optimal assignment and the related optimal $\pi(i)$ for each node i of the graph,
  IF $\pi(i) + c_{i,j}* \geq \pi(j) \quad \forall (i,j) \notin S_k$, THEN
  the assignment is optimal for the complete bipartite graph.

- Once an optimal assignment is reached for shortlist S_k , an iterative approach can be applied where
  EITHER the value of k is increased ($k = k + 1$)
  OR the optimality test is performed
  UNTIL global optimality is reached.

Primal approach with cycle canceling for AP

Practical computational considerations

- Bellman's algorithm requires at most n iterations where each iteration takes $O(m)$ time: here m denotes the cardinality of the arc set.
- Correspondingly, the smallest is the shortlist size, the fastest is the iteration: for low values of k, we have $O(m) \approx O(n)$.
- Once a Bellman's iteration is executed, it is possible to check in $O(n)$ time the presence of a negative cycle on an auxiliary graph with the same nodes and at most n arcs (arc (i, j) exists in the auxiliary graph if vertex i has been detected as predecessor of vertex j).
- Typically, one or two iterations are sufficient to detect a negative length cycle. Correspondingly, in most cases, the practical complexity for detecting a cycle is $O(n)$.

Preliminary computational results

- We compared the proposed approach denoted PCCS (Primal Cycle Canceling with Shortlist) with CSA and IIO.
- All the experiments were performed on a PC equipped with a CPU 12th Gen Intel Core i9-12900 3.8GHz, 32GB of RAM, and running Ubuntu 22.04 LTS.
- In this preliminary testing, the cost coefficients were drawn from the discrete uniform distribution $U\{1, K\}$.
- The initial solution is a standard row least cost greedy approach.
- We present here the results on dense instances with n sources and destinations where $n = 1000, 2000, 4000, 8000, 16000$ and $K = n$.
- Ten instances were considered for each value of n.

Preliminary computational results

- We report for PCCS the number of iterations (IT) namely the number of detected negative cycles, the number of calls to the Bellman's function (BF) and the CPU time in seconds (CPU).
- For IIO and CSA, the CPU time is reported.

Size	PCCS			IIO	CSA
$n =$	IT	BF	CPU	CPU	CPU
1000	111	365	0.03	0.05	0.02
2000	192	659	0.14	0.17	0.09
4000	335	1134	0.48	0.64	0.41
8000	600	2040	1.86	2.51	1.79
16000	1070	3582	7.01	10.347	7.82

Conclusions

- We have proposed a primal cycle canceling approach with shortlist for AP.
- In this preliminary testing with uniform distribution, the approach is competitive with respect to CSA.
- On these instances, negative cycles are typically found in linear time and a practical $O(n^2)$ time complexity experimentally appears.
- A peculiar behavior is encountered: typically a negative cycle is detected in the first or second Bellman's iteration.
- Also, the choice of the shortlist allows most of the time to deal with a very sparse graph inducing a strong reduction in the CPU time.
- Still, further testing on other instances distributions need to be considered and other competing algorithms as well...