

Linear programming and duality theory

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Linear Programming (LP)

A *linear program* is defined by

- linear constraints,
- a linear objective function.

Its variables are continuous and they can be

- free (unrestricted in sign),
- non-negative (constrained to be non-negative).

LP: general form

In its general form, it reads like this:

$$\begin{aligned} P) \text{ minimize } & z = c^T x \\ \text{subject to } & a'x \geq b' \\ & a''x = b'' \\ & x' \geq 0 \\ & x'' \text{ free} \end{aligned}$$

We can always get rid of the **free variables** by replacing each of them with the difference of two non-negative variables.

We can always get rid of **linear equalities** by variable elimination.

LP: inequalities form

In its inequalities form, a linear program reads like this:

$$\begin{aligned} P) \text{ minimize } & z = c^T x \\ \text{subject to } & ax \geq b \\ & x \geq 0 \end{aligned}$$

Every solution of the problem is a point in an n -dimensional space, where n is the number of variables.

Solutions are *feasible* if they comply with all the constraints and the non-negativity restrictions. Otherwise they are *infeasible*.

The set of feasible solutions is the *feasible region* of the linear program.

Constraints: geometrical interpretation

The constraints set of a linear program is made of linear inequalities.

In the variables space

- linear equalities correspond to *hyperplanes*;
- linear inequalities correspond to *halfspaces*.

The feasible region is a system of linear inequalities; it corresponds to the *intersection of halfspaces*, i.e. to a *polyhedron*.

Half spaces are convex; hence their intersection is also convex:
polyhedra are convex.

Theorem (Minkowsky and Weil). Every point in a polyhedron can be obtained as a convex combination of its **extreme points** and its **extreme rays**.

Polyhedra: three cases

Given a polyhedron one of these three cases occurs:

- the polyhedron is not empty and it is bounded (polytope);
- the polyhedron is not empty and it is not bounded;
- the polyhedron is empty.

Case 1: polytope

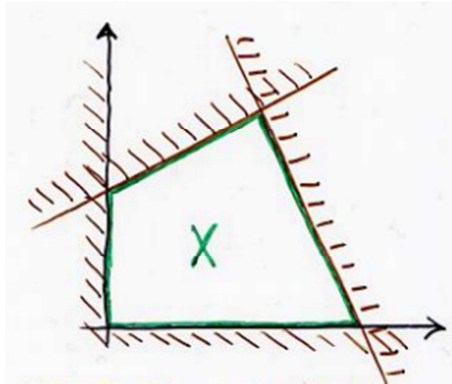


Figure: A polytope is defined by its extreme points.

Case 2: unbounded polyhedron

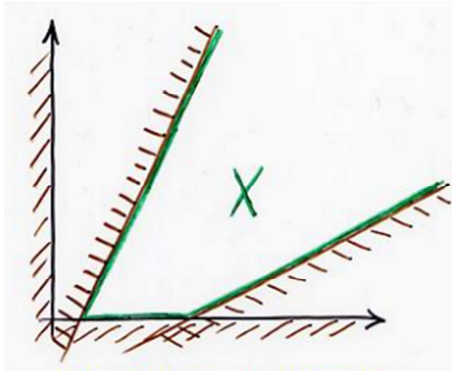


Figure: An unbounded polyhedron is defined by its extreme points and its extreme rays.

Case 3: empty polyhedron

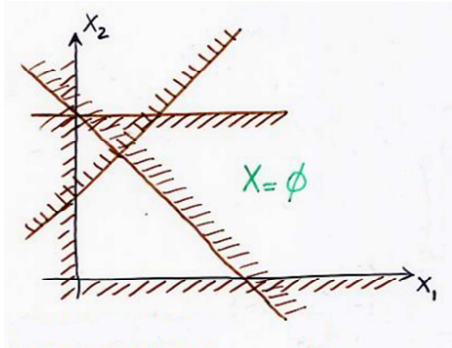


Figure: An empty polyhedron contains no points at all.

Objective function: geometrical interpretation

The objective function of a linear program is represented by parallel hyperplanes, such that points on the same hyperplane have the same value.

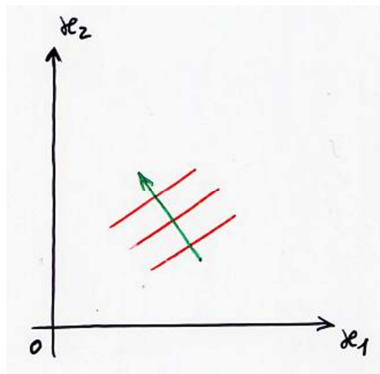


Figure: minimize $z = 2x_1 - 3x_2$

Linear programs: three cases

Given a linear program P , one of these three cases occurs (and the simplex algorithm detects it in a finite number of steps):

- P has a finite optimal solution;
- P is unbounded;
- P is infeasible.

Case 1: finite optimal solution

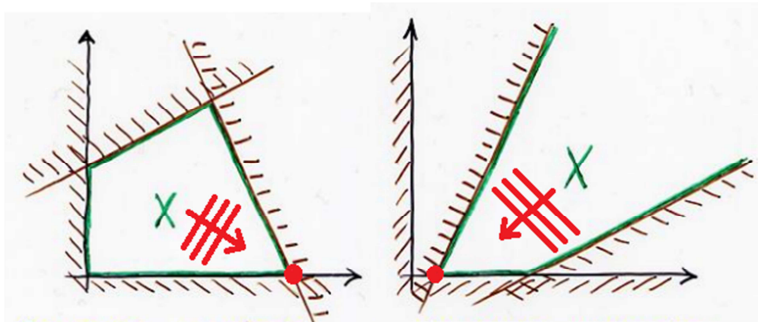


Figure: Two cases in which the linear program has a finite optimal solution.

Case 2: unbounded problem

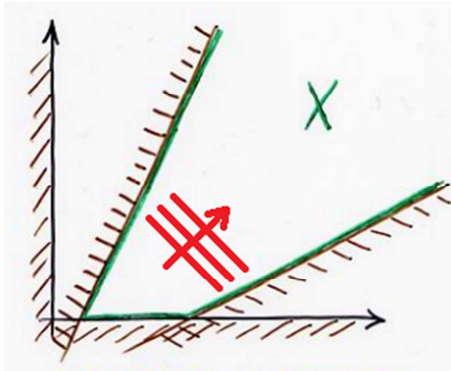


Figure: The linear program is unbounded if the direction of the objective function is in the cone defined by the extreme rays.

Case 3: infeasible problem

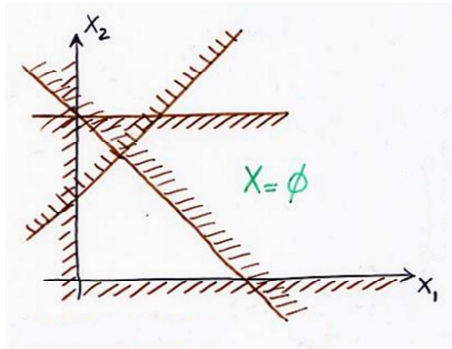


Figure: Empty polyhedron: the linear program is infeasible, independently of the objective function.

LP: standard form

A linear program is in its standard form, when all inequalities have been turned into equalities by explicitly inserting non-negative *slack variables* or *surplus variables*.

Inequalities form:

$$\begin{aligned} P) \text{ minimize } z &= c^T x \\ \text{subject to } a'x &\geq b' \\ a''x &\leq b'' \\ x &\geq 0. \end{aligned}$$

m inequality constraints
 n non-negative variables.

Standard form:

$$\begin{aligned} P) \text{ minimize } z &= c^T x \\ \text{subject to } a'x - x^{surplus} &= b' \\ a''x + x^{slack} &= b'' \\ x, x^{surplus}, x^{slack} &\geq 0. \end{aligned}$$

m equality constraints
 $n + m$ non-negative variables.

The number of inequalities is the same in the two models.

Bases

A *base* is a subset of m linearly independent columns of the constraint matrix.

The m variables corresponding to the columns of the base are *basic*.

The other n variables are *non-basic*.

If the value of the non-basic variables is fixed, we are left with a non-degenerate system of m linear equalities and m variables, which provides a unique solution x .

If $x \geq 0$, then it is feasible. Otherwise it is infeasible.

Base solutions

An inequality constraint is *active* iff its slack/surplus variable is null.

If we fix n (non-basic) variables to 0, we define a solution x in which n constraints are active. The points of intersection of n constraints in an n -dimensional variable space are called *base solutions*. They can be feasible or infeasible.

All the extreme points of the polyhedron are feasible base solutions.

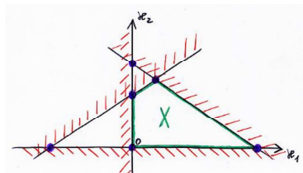


Figure: An example with $n = 2$, $m = 2$. There are 6 base solutions: 4 are feasible, 2 are infeasible.

Linear Programming (LP) duality

Every linear program P has a dual linear program D .

$$P) \min z = c'^T x' + c''^T x''$$

$$\text{s.t. } a'x' + a''x'' \geq b' \quad [y']$$

$$d'x' + d''x'' = b'' \quad [y'']$$

$$x' \geq 0$$

$$x'' \text{ free}$$

$$D) \max w = b'^T y' + b''^T y''$$

$$\text{s.t. } a'^T y' + d'^T y'' \leq c' \quad [x']$$

$$a''^T y' + d''^T y'' = c'' \quad [x'']$$

$$y' \geq 0$$

$$y'' \text{ free}$$

The fundamental theorem of LP duality

Given a primal-dual pair, one of these four cases occurs (and the simplex algorithm detects it in a finite number of steps):

- both P and D have a finite optimal solution;
- P is unbounded and D is infeasible;
- D is unbounded and P is infeasible;
- both P and D are infeasible.

Weak duality theorem

Weak duality theorem.

For each feasible solution x of P and for each feasible solution y of D , $z(x) \geq w(y)$.

Corollary 1.

If P is unbounded, then D is infeasible.

Corollary 2.

If x is feasible for P and y is feasible for D and $z(x) = w(y)$, then both x and y are also optimal.

Strong duality theorem

Strong duality theorem.

If there exist a feasible and optimal solution x^* for P and a feasible and optimal solution y^* for D , then $z(x^*) = w(y^*)$.

Complementary slackness theorem

Complementary slackness theorem.

Given a feasible solution x for P and a feasible solution y for D , necessary and sufficient condition for them to be optimal is:

- Primal complementary slackness conditions (they are n):

$$x' (c' - a'^T y' - d'^T y'') = 0$$

- Dual complementary slackness conditions (they are m):

$$y' (a' x' + a'' x'' - b') = 0.$$

They only refer to inequality constraints, because those corresponding to equality constraints are always trivially satisfied.

Primal-dual algorithms

A primal-dual algorithm solves linear programming problems exploiting duality theory and in particular the CSCs.

The algorithm is initialized with a **dual feasible** solution and a corresponding **primal solution** (in general, infeasible) satisfying the CSCs.

After every iteration the algorithm keeps a pair of **primal (infeasible)** and **dual (feasible)** solutions, satisfying the CSCs.

The algorithm alternates two types of iterations, and it monotonically decreases **primal infeasibility** until it achieves **primal feasibility**.

- **Primal iteration**: keeping the current **dual feasible solution** fixed, find a **primal solution** minimizing **primal infeasibility** among those satisfying the CSCs;
- **Dual iteration**: keeping the current **primal solution** fixed, modify the **dual solution**, keeping it **feasible** and the CSCs satisfied.