Interior point methods

Discrete Optimization

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Based on: J. Gonzio, *Interior point methods for linear programming*, NATCOR, Edinburgh, 2014.

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Simplex algorithm vs Interior point methods

The worst-case computational complexity of the simplex algorithm is exponential.

V. Klee, G. Minty, *How good is the simplex algorithm?*, Inequalities, O. Shisha ed., Academic Press (1972) 159-175.

The worst-case computational complexity of interior point methods can be polynomial.

N. Karmarkar, *A new polynomial time algorithm for LP*, Combinatorica 4 (1984) 373-395.

Main components

Main components of interior point methods:

- LP duality theory
- Lagrangean function
- first order optimality conditions for NLP
- logarithmic barriers
- Newton method

Optimality conditions

$$
\begin{array}{llll}
P) & \text{minimize } z = c^T x & D) & \text{maximize } w = b^T y \\
 & \text{s.t. } Ax = b & & \text{s.t. } A^T y + s = c \\
 & \quad x \geq 0 & & \quad s \geq 0
\end{array}
$$

Lagrangean function: $L(x, y) = c^T x - y^T(Ax - b) - s^T x$.

Optimality conditions:

$$
\begin{aligned}
&Ax = b \\
&A^T y + s = c \\
&x_j s_j = 0 \ \ \forall j \\
&(x, s) \geq 0
\end{aligned}
$$

Logarithmic barrier

Each inequality $x_j \geq 0$ is replaced by a penalty term $-\log x_j$.

Minimizing

$$-\sum_j \log x_j$$

is equivalent to maximizing

$$\prod_j x_j.$$

Effect: it prevents x variables from approaching 0 making the constraints $x_j \geq 0$ active.

Logarithmic barrier

The primal LP

$$
\begin{array}{lll}
P) & \text{minimize } z = & c^T x \\
 & \text{s.t.} & Ax = b \\
 & & x \geq 0
\end{array}
$$

is replaced by the primal barrier program

$$
\begin{array}{ll}
\text{minimize} & c^T x - \mu \sum_j \log x_j \\
\text{s.t.} & Ax = b
\end{array}
$$

Lagrangean function: $L(x, y, \mu) = c^T x - y^T (Ax - b) - \mu \sum_j \log x_j$.

Optimality conditions

Conditions for a stationary point:

$$\begin{aligned} \nabla_x L(x, y, \mu) = \quad & c - A^T y - \mu \sum_j \frac{1}{x_j} & = 0 \\ \nabla_y L(x, y, \mu) = \quad & Ax - b & = 0 \end{aligned}$$

First order optimality conditions:

$$\begin{aligned} & Ax = b \\ & A^T y + s = c \\ & x_j s_j = \mu \quad \forall j \\ & (x, s) > 0 \end{aligned}$$

Complementarity

Complementarity conditions: $x_j s_j = 0 \quad \forall j$.

Simplex algorithm: some variables are set to 0.

- Basic variables: $x_j s_j = 0$ because $s_j = 0$.
- Non-basic variables: $x_j s_j = 0$ because $x_j = 0$.

Interior point methods: no variable is set to 0.

$$x_j s_j = \mu \quad (x, s) > 0$$

and μ is iteratively forced to 0.

Parameter μ controls the distance from optimality:

$$c^T x - b^T y = c^T x - x^T A^T y = x^T(c - A^T y) = x^T s = n\mu.$$

Newton method

Analytic center (μ-center): a point (x, y, s) that satisfies the first-order optimality conditions.

The sequence of μ-centers is the primal-dual central trajectory.

Newton method finds a solution of $f(x) = 0$.

A tangent line to $f(x)$ in $(x^{(k)}, f(x^{(k)}))$ has equation

$$z - f(x^{(k)}) = \nabla f(x^{(k)})(x - x^{(k)}).$$

Setting $z = 0$, we find a new point

$$x^{(k+1)} = x^{(k)} - (\nabla f(x^{(k)}))^{-1} f(x^{(k)}).$$

Newton method

In an interior point method

$$f(x,y,s) = \begin{bmatrix} Ax - b \\ A^T y + s - c \\ XSe - \mu e \end{bmatrix}$$

where $X = diag\{x_1, \ldots, x_n\}$, $S = diag\{s_1, \ldots, s_n\}$, $e = (1, \ldots, 1)$.

$$\nabla f(x,y,s) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix}$$

The Newton direction $(\Delta x, \Delta y, \Delta s)$ in (x, y, s) is found by solving a linear system:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - s \\ \mu e - XSe \end{bmatrix}$$

Newton method

Idea: make only one Newton step and then reduce μ.

Select $(x^0, y^0, s^0) \in F^0$
$k \leftarrow 0$
$\mu_0 \leftarrow \frac{1}{n}(x^0)^T s^0$
$\alpha_0 \leftarrow 0.999$
Select $\sigma \in (0, 1)$
repeat
  $k \leftarrow k + 1$
  $\mu_k \leftarrow \sigma \mu_{k-1}$
  Compute $(\Delta x, \Delta y, \Delta s)$
  $\alpha_P \leftarrow \max\{\alpha > 0 : x + \alpha \Delta x \geq 0\}$
  $\alpha_D \leftarrow \max\{\alpha > 0 : s + \alpha \Delta s \geq 0\}$
  $x^{k+1} \leftarrow x^k + \alpha_0 \alpha_P \Delta x$
  $y^{k+1} \leftarrow y^k + \alpha_0 \alpha_D \Delta y$
  $s^{k+1} \leftarrow s^k + \alpha_0 \alpha_D \Delta s$
until optimality
