The Miller-Tucker-Zemlin formulation for the Asymmetric TSP

Discrete Optimization

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The ATSP

Given a digraph $D = (N, A)$ and a cost function $c : A \mapsto \Re$, the Asymmetrtic Traveling Salesman Problem (ATSP) is the problem of finding a minimum cost Hamiltonian circuit in D:

$$
\begin{aligned}
\text{minimize } z = & \sum_{(i,j)\in A} c_{ij}x_{ij} & & \\
\text{s.t. } & \sum_{(i,j)\in A} x_{ij} = 1 & \forall i \in N & \\
& \sum_{(i,j)\in A} x_{ij} = 1 & \forall j \in N & \\
& S.E.C & & (1) \\
& x_{ij} \in \{0, 1\} & \forall (i,j) \in A &
\end{aligned}
$$

Constraints (1) are called subtour elimination constraints.

W.l.o.g: node 1 is the first and last node in the circuit (depot).

Define $N' = N \backslash \{1\}$.

DFJ and circuit S.E.C.

Dantig-Fulkerson-Johnson subtour elimination constraints:

$$\text{DFJ)} \quad \sum_{(i,j)\in A(S)} x_{ij} \leq |S| - 1 \quad \forall S \subset N', |S| \geq 2 \tag{2}$$

where $A(S)$ is the arc set of the subgraph induced by $S \subseteq N'$.

Circuit subtour elimination constraints:

$$\sum_{(i,j)\in C} x_{ij} \leq |C| - 1 \quad \forall C \subset A(N') \tag{3}$$

where C is a circuit in $A(N')$.

“Natural” formulations: x variables, exponential size, tight bounds.

“Extended” formulations: more vars, polynomial size, loose bounds.

Miller-Tucker-Zemlin S.E.C. (1960)

$$u_i - u_j + (n-1)x_{ij} \leq n-2 \qquad \forall i \neq j \in N' \tag{4}$$
$$u_i \in \Re \qquad \forall i \in N' \tag{5}$$

Constraints (4) imply that

$$x_{ij} \Rightarrow u_j \geq u_i + 1.$$

W.l.o.g.:

$$0 \leq u_i \leq n-2.$$

Meaning of u_i: n. of nodes between 1 and i.

Projecting (4)-(5) on the x subspace, one gets the polyhedron defined by

$$\sum_{(i,j)\in C} x_{ij} \leq |C| - \frac{|C|}{n-1} \qquad \forall C \subset A(N')$$

which are weaker than the Circuit s.e.c.

Improved MTZ SECs (Desrochers, Laporte, 1991)

Lift x_{ji} in (4):

$$u_i - u_j + (n-1)x_{ij} + \alpha x_{ji} \leq n-2 \quad \forall i \neq j \in N'.$$

$$x_{ji} = 1 \Rightarrow \begin{cases} x_{ij} = 0 \\ u_i = u_j + 1. \end{cases}$$

$$\begin{aligned} & u_i - u_j + (n-1)x_{ij} + \alpha x_{ji} \leq n-2 \\ & 1 + 0 + \alpha \leq n-2 \\ & \alpha^* = n-3. \end{aligned}$$

Lifted inequality:

$$DL(i,j): \quad u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} \leq n-2 \quad \forall i \neq j \in N'.$$

Improved MTZ SECs (Desrochers, Laporte, 1991)

$$DL(i,j): \quad u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} \leq n-2 \quad \forall i \neq j \in N'.$$

Remark 1.

$$x_{ij} = 1 \Rightarrow x_{ji} = 0 \Rightarrow \begin{cases} u_i - u_j + n - 1 \leq n-2 \\ u_j - u_i + n - 3 \leq n-2 \end{cases} \Rightarrow u_i - u_j = 1$$

Improved MTZ SECs (Desrochers, Laporte, 1991)

Remark 2. Summing the two inequalities

$$
\begin{aligned}
DL(i,j): &\quad u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} \leq n-2 \\
DL(j,i): &\quad u_j - u_i + (n-1)x_{ji} + (n-3)x_{ij} \leq n-2
\end{aligned}
$$

one obtains

$$
\begin{aligned}
&(2n-4)x_{ij} + (2n-4)x_{ji} \leq 2n-4 \\
&x_{ij} + x_{ji} \leq 1
\end{aligned}
$$

i.e. the DFJ s.e.c. for $|S| = 2$.

Cuts on three nodes (Bektas, Gouveia, 2014)

Summing up

$$
\begin{aligned}
DL(i,j):&\quad u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} \leq n-2\\
DL(j,k):&\quad u_k - u_j + (n-1)x_{kj} + (n-3)x_{jk} \leq n-2
\end{aligned}
$$

one obtains

$$u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{ji} + x_{kj}) \leq 2n-4.$$

Cuts on three nodes

Lifting x_{ik}:

$$u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{ji} + x_{kj}) + \alpha x_{ik} \leq 2n - 4.$$

$$x_{ik} = 1 \quad \Rightarrow \quad \begin{cases} x_{ij} = 0 \\ u_k = u_i + 1 \\ x_{ji} + x_{kj} \leq 1 \end{cases}$$

$$\begin{aligned} -1 + (n-3)(x_{ji} + x_{kj}) + \alpha &\leq 2n - 4 \\ \alpha &\leq 2n - 3 - (n-3)(x_{ji} + x_{kj}) \\ \alpha &\leq 2n - 3 - (n-3) \\ \alpha^* &= n \end{aligned}$$

Lifted inequality:

$$u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{ji} + x_{kj}) + n x_{ik} \leq 2n - 4.$$

Cuts on three nodes

Lifting x_{ki}:

$$u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{ji} + x_{kj}) + nx_{ik} + \beta x_{ki} \leq 2n - 4.$$

$$x_{ki} = 1 \quad \Rightarrow \quad \begin{cases} u_i = u_k + 1 \\ x_{ij} + x_{jk} \leq 1 \\ x_{ji} = 0 \\ x_{kj} = 0 \\ x_{ik} = 0 \end{cases}$$

$$\begin{aligned} & 1 + (n-1)(x_{ij} + x_{jk}) + \beta \leq 2n - 4 \\ & \beta \leq 2n - 5 - (n-1)(x_{ij} + x_{jk}) \\ & \beta \leq 2n - 5 - (n-1) \\ & \beta^* = n - 4 \end{aligned}$$

Lifted inequality $BG(i, j, k)$:

$$u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{ji} + x_{kj}) + nx_{ik} + (n-4)x_{ki} \leq 2n - 4.$$

Cuts on three nodes

The lifted inequalities

$$u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{ji} + x_{kj}) + nx_{ik} + (n-4)x_{ki} \leq 2n - 4$$

are facet-defining for the polyhedron defined by the constraints that forbid cycles of order 3.

Summing up $BG(i,j,k)$ and $BG(k,j,i)$,

$$u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{ji} + x_{kj}) + nx_{ik} + (n-4)x_{ki} \leq 2n - 4$$
$$u_k - u_i + (n-1)(x_{kj} + x_{ji}) + (n-3)(x_{jk} + x_{ij}) + nx_{ki} + (n-4)x_{ik} \leq 2n - 4$$

one obtains

$$(2n-4)(x_{ij} + x_{jk}) + (2n-4)(x_{ji} + x_{kj}) + (2n-4)(x_{ik} + x_{ki}) \leq 4n - 8$$
$$x_{ij} + x_{jk} + x_{ji} + x_{kj} + x_{ik} + x_{ki} \leq 2$$

that is the DFJ s.e.c. for $|S| = 3$.

More cuts for three nodes

Summing up $BG(i, j, k)$ and $BG(i, k, j)$,

$$u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (n-3)(x_{ji} + x_{kj}) + nx_{ik} + (n-4)x_{ki} \leq 2n - 4$$
$$u_i - u_j + (n-1)(x_{ik} + x_{kj}) + (n-3)(x_{ki} + x_{jk}) + nx_{ij} + (n-4)x_{ji} \leq 2n - 4$$

one obtains

$$2u_i - u_j - u_k + (2n-2)(x_{ij} + x_{ik}) + (2n-4)(x_{jk} + x_{kj}) + (2n-7)(x_{ji} + x_{ki}) \leq 4n - 8$$

but this inequality is dominated by

$$2u_i - u_j - u_k + (2n-2)(x_{ij} + x_{ik}) + (2n-5)(x_{jk} + x_{kj}) + (2n-8)(x_{ji} + x_{ki}) \leq 4n - 10$$

which is valid.

Symmetrically, a valid inequality is

$$-2u_i + u_j + u_k + (2n-8)(x_{ij} + x_{ik}) + (2n-5)(x_{jk} + x_{kj}) + (2n-2)(x_{ji} + x_{ki}) \leq 4n - 10.$$

Proof of validity

$$2u_i - u_j - u_k + (2n-2)(x_{ij} + x_{ik}) + (2n-5)(x_{jk} + x_{kj}) + (2n-8)(x_{ji} + x_{ki}) \leq 4n - 10$$

When $x_{ij} = x_{jk} = 1$ or $x_{ik} = x_{kj} = 1$,

$$2u_i - u_j - u_k + 2n - 2 + 2n - 5 \leq 4n - 10$$
$$u_j + u_k \geq 2u_i + 3.$$

When $x_{ji} = x_{ik} = 1$ or $x_{ki} = x_{ij} = 1$,

$$2u_i - u_j - u_k + 2n - 2 + 2n - 8 \leq 4n - 10$$
$$(u_j - u_j) + (u_i - u_k) \leq 0.$$

When $x_{jk} = x_{ki} = 1$ or $x_{kj} = x_{ji} = 1$,

$$2u_i - u_j - u_k + 2n - 5 + 2n - 8 \leq 4n - 10$$
$$u_j + u_k \geq 2u_i - 3.$$

When one variable $x_{ji} = 1$ or $x_{ki} = 1$ and the other five variables are 0,

$$2u_i - u_j - u_k + 2n - 8 \leq 4n - 10$$
$$(u_i - u_j) + (u_i - u_k) \leq 2(n-1).$$

2-path constraints (1/2)

Summing up $MTZ(i,j)$ and $MTZ(j,k)$,

$$u_i - u_j + (n-1)x_{ij} \leq n-2$$
$$u_j - u_k + (n-1)x_{jk} \leq n-2$$

one obtains $u_i - u_k + (n-1)(x_{ij} + x_{jk}) \leq 2n-4$.

Lift x_{ik}: $u_i - u_k + (n-1)(x_{ij} + x_{jk}) + \alpha x_{ik} \leq 2n-4$.

$$x_{ik} = 1 \quad \Rightarrow \quad \begin{cases} u_k = u_i + 1 \\ x_{ij} = x_{jk} = 0 \end{cases}$$

$-1 + 0 + \alpha \leq 2n-4 \quad \Rightarrow \quad \alpha^* = 2n-3$.

Lift x_{ki}: $u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (2n-3)x_{ik} + \beta x_{ki} \leq 2n-4$.

$$x_{ki} = 1 \quad \Rightarrow \quad \begin{cases} u_i = u_k + 1 \\ x_{ij} + x_{jk} \leq 1 \\ x_{ik} = 0 \end{cases}$$

$1 + (n-1) + 0 + \beta \leq 2n-4 \quad \Rightarrow \quad \beta^* = n-4$.

$$A) \quad u_i - u_k + (n-1)(x_{ij} + x_{jk}) + (2n-3)x_{ik} + (n-4)x_{ki} \leq 2n-4$$

2-path constraints (2/2)

Similarly one can obtain

$$B) \quad u_k - u_i + (n-4)(x_{ij} + x_{jk}) + (2n-7)x_{ik} + (n-1)x_{ki} \leq 2n-4$$

(No proof provided).

Lifting x_{ik} in the circuit inequality $x_{ij} + x_{jk} + x_{ki} \leq 2$, one obtains

$$2x_{ik} + x_{ij} + x_{jk} + x_{ki} \leq 2$$

Summing up *A* and *B* inequalities, one obtains

$$(4n+10)x_{ik} + (2n+5)(x_{ij} + x_{jk} + x_{ki}) \leq 4n-10$$

that are the same.

No method is known to obtain inequalities for the polyhedron defined by the constraints forbidding cycles or order 4 (or more).

Inequalities from bounds

From lower bounds $u_i \geq 0 \ \forall i \in N'$:

Lift $(1 - x_{1i})$: $u_i \geq \alpha(1 - x_{1i})$.

$x_{1i} = 0 \quad \Rightarrow \quad u_i = 1.$

$1 \geq \alpha(1 - 0) \quad \Rightarrow \quad \alpha^* = 1.$

Lift x_{i1}: $u_i + x_{1i} \geq 1 + \beta x_{i1}$.

$$x_{i1} = 1 \quad \Rightarrow \begin{cases} u_i = n - 2 \\ x_{1i} = 0 \end{cases}$$

$n - 2 + 0 \geq 1 + \beta \quad \Rightarrow \quad \beta^* = n - 3.$

Lifted inequality:

$$u_i + x_{1i} \geq 1 + (n - 3)x_{i1} \quad \forall i \in N'.$$

Inequalities from bounds

From upper bounds $u_i \leq n-2 \; \forall i \in N'$:

Lift $(1-x_{i1})$: $u_i \leq \alpha(1-x_{i1})$.

$x_{i1} = 0 \;\; \Rightarrow \;\; u_i \leq n-3$.

$n-3+\alpha \leq n-2 \;\; \Rightarrow \;\; \alpha^* = 1$.

Lift x_{i1}: $u_i + (1-x_{i1}) + \beta x_{1i} \leq n-2$.

$$x_{1i} = 1 \;\; \Rightarrow \begin{cases} u_i = 0 \\ x_{i1} = 0 \end{cases}$$

$0+1+\beta \leq n-2 \;\; \Rightarrow \;\; \beta^* = n-3$.

Lifted inequality:

$$u_i + (1-x_{i1}) + (n-3)x_{1i} \leq n-2 \quad \forall i \in N'.$$

Inequalities from bounds (Sawik, 2016)

From lower bounds $u_i \geq 0 \; \forall i \in N'$:

Lift x_{i1}: $u_i \geq \gamma x_{i1}$.

$x_{1i} = 1 \;\; \Rightarrow \;\; u_i = n - 2$.

$n - 2 \geq \gamma \;\; \Rightarrow \;\; \gamma^* = n - 2$.

Lifted inequality:

$$u_i \geq (n-2) x_{i1} \quad \forall i \in N'.$$

From upper bounds $u_i \leq n - 2 \; \forall i \in N'$:

Lift x_{1i}: $u_i + \delta x_{1i} \leq n - 2$.

$x_{1i} = 1 \;\; \Rightarrow \;\; u_i = 0$.

$0 + \delta \leq n - 2 \;\; \Rightarrow \;\; \delta^* = n - 2$.

Lifted inequality:

$$u_i + (n-2) x_{1i} \leq n - 2 \quad \forall i \in N'.$$