# Extended formulations for the ATSP (with precedence constraints)

## Discrete Optimization

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# The ATSP

Given a digraph $D = (N, A)$ and a cost function $c : A \mapsto \Re$, the Asymmetric Traveling Salesman Problem (ATSP) is to find a minimum cost Hamiltonian circuit.

Extended formulations: additional variables, polynomial size, loose bounds.

ATSP with precedence constraints (ATSPPC):

- a depot in $N$ is specified;
- a set of precedecessors $P_i \subseteq N$ is given for each node $i \in N$.

# An extended formulation (Gouveia, Pires, 1999, 2001)

$$
\begin{aligned}
\text{minimize } z = & \sum_{(i,j)\in A} c_{ij}x_{ij} & \\
\text{s.t. } & \sum_{(i,j)\in A} x_{ij} = 1 & \forall i \in N \\
& \sum_{(i,j)\in A} x_{ij} = 1 & \forall j \in N \\
& x_{ij} \leq y_{ij} & \forall i \neq j \in N' \\
& x_{ij} + y_{ij} \leq 1 & \forall i \neq j \in N' \\
\text{GP(i,j,k)} \quad & x_{ij} + y_{ki} \leq y_{kj} + 1 & \forall i \neq j \neq k \in N' \\
& x_{ij} \in \{0,1\} & \forall (i,j) \in A \\
& y_{ij} \geq 0 & \forall i \neq j \in N'
\end{aligned}
$$

Depot: node 1. Subset $N' = N \setminus \{1\}$.
Arc variables $x_{ij}$: arc $(i,j)$ is traversed.
Precedence variables $y_{ij}$: node $i \in N'$ is visited before node $j \in N'$.

# Lifting the inequalities

Lift $x_{ji}$ in $GP(i,j,k)$: $x_{ij} + y_{ki} + \alpha x_{ji} \leq y_{kj} + 1$.

$$x_{ji} = 1 \quad \Rightarrow \begin{cases} x_{ij} = 0 \\ y_{ki} = y_{kj} \end{cases} .$$

$$0 + y_{ki} + \alpha \leq y_{kj} + 1 \quad \Rightarrow \quad \alpha^* = 1.$$

Lifted inequality:

$$\text{LGP1(i,j,k):} \quad x_{ij} + y_{ki} + x_{ji} \leq y_{kj} + 1.$$

# Lifting the inequalities

Lift $x_{kj}$ in $GP(i,j,k)$: $x_{ij} + y_{ki} + \beta x_{kj} \leq y_{kj} + 1$.

$$x_{kj} = 1 \quad \Rightarrow \left\{ \begin{array}{l} x_{ij} = 0 \\ y_{kj} = 1 \\ y_{ki} \leq 1 \end{array} \right\}.$$

$$0 + y_{ki} + \beta \leq 1 + 1 \quad \Rightarrow \quad \beta^* = 1.$$

Lift $x_{ik}$: $x_{ij} + y_{ki} + x_{kj} + \gamma x_{ik} \leq y_{kj} + 1$.

$$x_{ik} = 1 \quad \Rightarrow \left\{ \begin{array}{l} x_{ij} = y_{ki} = 0 \\ x_{kj} \leq y_{kj} \end{array} \right\}.$$

$$0 + 0 + x_{kj} + \gamma \leq y_{ky} + 1 \quad \Rightarrow \quad \gamma^* = 1.$$

Lifted inequality:

LGP2(i,j,k) $\quad x_{ij} + x_{kj} + x_{ik} + y_{ki} \leq y_{kj} + 1.$

# An extended formulation (Sarin et al., 2005)

$$\text{minimize } z = \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\text{s.t.} \sum_{(i,j)\in A} x_{ij} = 1 \qquad \forall i \in N$$

$$\sum_{(i,j)\in A} x_{ij} = 1 \qquad \forall j \in N$$

$$x_{ij} \leq y_{ij} \qquad \forall i \neq j \in N'$$

$$y_{ij} + y_{ij} \leq 1 \qquad \forall i \neq j \in N'$$

$$\text{S3(i,j,k)} \quad y_{ij} + y_{jk} + y_{ki} \leq 2 \qquad \forall i \neq j \neq k \in N'$$

$$x_{ij} \in \{0, 1\} \qquad \forall (i,j) \in A$$

$$y_{ij} \geq 0 \qquad \forall i \neq j \in N'$$

Further constraints: $x_{1i} + x_{i1} \leq 1 \quad \forall i \in N'$.

# An extended formulation (Sarin et al., 2005)

Since $y_{jk} + y_{kj} = 1$,

$$GP(i,j,k): \quad x_{ij} + y_{ki} \leq y_{kj} + 1 \quad \Leftrightarrow \quad y_{ki} + x_{ij} + y_{jk} \leq 2.$$

$$LGP1(i,j,k): \quad x_{ij} + y_{ki} + x_{ji} \leq y_{kj} + 1 \quad \Leftrightarrow \quad y_{ki} + x_{ij} + y_{jk} + x_{ji} \leq 2.$$

The non-lifted inequality $GP(i,j,k)$ is dominated by the non-lifted inequality

$$S3(i,j,k): \quad y_{ki} + y_{ij} + y_{jk} \leq 2$$

since $x_{ij} \leq y_{ij}$.

# An extended formulation (Sarin et al., 2005)

Lift $x_{ji}$ in $S3(i,j,k):$ $\quad : y_{ki} + y_{ij} + y_{jk} + \alpha x_{ji} \leq 2$.

$$x_{ji} = 1 \quad \Rightarrow \begin{cases} y_{ji} = 1 \\ y_{ij} = 0 \\ y_{jk} + y_{ki} \leq 1 \end{cases} .$$

$$y_{ki} + 0 + y_{jk} + \alpha \leq 2 \quad \Rightarrow \quad \alpha^* = 1.$$

The lifted inequality:

$$\text{LS3(i,j,k)} \quad y_{ij} + y_{jk} + y_{ki} + x_{ji} \leq 2$$

dominates $LGP1(i,j,k):$ $\quad x_{ij} + y_{jk} + y_{ki} + x_{ji} \leq 2$.

Symmetrically, one obtains

$$y_{ij} + y_{jk} + y_{ki} + x_{kj} \leq 2$$

$$y_{ij} + y_{jk} + y_{ki} + x_{ik} \leq 2$$