Minimum mean cycle problem

Combinatorial optimization

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Problem definition

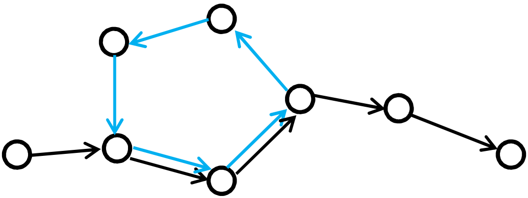
We are given a strongly connected digraph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ with a cost function $c : \mathcal{A} \mapsto \Re$.

The arc costs are unrestricted in sign; negative cost cycles may exist.

We want to find the cycle of minimum mean cost.
The mean cost of a cycle is the average value of costs of the arcs in the cycle.

Directed walks

A directed walk on a digraph is a sequence of nodes, including possible repetitions as well as cycles.



Every directed walk is composed by a directed path and possibly some directed cycles.

The algorithm

Let $s \in \mathcal{N}$ be a node.

Let $d^k(j)$ be the cost of the minimum cost directed walk from s to j with k arcs.

$$\begin{cases} d^0(j) = \infty & \forall j \in \mathcal{N} \\ d^k(j) = \min_{(i,j) \in \mathcal{A}} \{d^{k-1}(i) + c(i,j)\} & \forall j \in \mathcal{N} \end{cases}$$

From the values $d^k(j)$ for all nodes one computes the values $d^{k+1}(j)$

for all nodes in $O(m)$.

Since $k = 0, \ldots, n$, the overall time complexity is $O(mn)$.

The optimal solution

Let μ^* be the mean cost of the minimum mean cost cycle.

Then

$$\mu^* = \min_{j \in \mathcal{N}} \max_{0 \leq k \leq n-1} \{ \frac{d^n(j) - d^k(j)}{n-k} \}$$

Therefore μ^* can be computed with the above labelling algorithm in $O(mn)$.

Now we must prove this statement on μ^* in two cases: when $\mu^* = 0$ and when $\mu^* \neq 0$.

Proof: $\mu^* \neq 0$

The main part of the proof concerns the case when both the lhs and the rhs are zero, i.e. when the digraph contains a cycle of zero cost and no negative cost cycles.

When $\mu^* \neq 0$, consider what happens if all arc costs are decreased by an amount Δ (unrestricted in sign):

- The optimum μ^* (the lhs) is reduced by Δ.
- Every value $d^k(j)$ is decreased by $k\Delta$.
- The rhs $\frac{d^n(j)-d^k(j)}{n-k}$ is decreased by $\frac{n\Delta-k\Delta}{n-k}$, that is by Δ.

Then both the lhs and the rhs are reduced by the same amount Δ.

Then, selecting $\Delta = \mu^*$ we prove that the statement remains true, if it is true for $\mu^* = 0$.

Proof: $\mu^* = 0$ (1)

We can assume now that $\mu^* = 0$.

Then

- the digraph contains a cycle C with zero cost (the optimal solution);
- the digraph does not contain any cycle with negative cost.

Since the digraph does not contain negative cost cycles, the shortest path problem is well defined.

Let $d(j)$ be the cost of the shortest path from s to j for each $j \in \mathcal{N}$.

Proof: $\mu^* = 0$ (2)

Compute the shortest path arborescence T_s from s to all the other nodes.

The reduced costs of the arcs are

$$\overline{c}(i,j) = c(i,j) + d(i) - d(j) \ \forall (i,j) \in \mathcal{A}.$$

For the properties of the shortest paths arborescence, the following relations hold:

- $\overline{c}(i,j) \geq 0 \quad \forall (i,j) \in \mathcal{A}$, because this is required by dual constraints.
- $\overline{c}(i,j) = 0 \quad \forall (i,j) \in C$, because the cost reductions $+d(i) - d(j)$ cancel out along cycles; therefore for each cycle the sum of the reduced costs is the same as the sum of the original costs.
- $\overline{c}(i,j) = 0 \quad \forall (i,j) \in T_s$, because of the complementary slackness conditions.

Proof: $\mu^* = 0$ (3)

Furthermore, the distances $\overline{d}^k(j)$ computed according to the reduced costs differ by a constant from the distances computed according to the original costs:

$$\overline{d}^k(j) - d^k(j) = \lambda_j \quad \forall k = 0, \ldots, n \;\; \forall j \in \mathcal{N}$$

because cost reductions $+d(i) - d(j)$ cancel out in the intermediate nodes along the path from s to j.
Therefore

$$d^n(j) - d^k(j) = \overline{d}^n(j) - \overline{d}^k(j) \quad \forall j \in \mathcal{N} \;\; \forall k = 0, \ldots, n.$$

Proof: $\mu^* = 0$ (4)

The shortest path from s to j has zero reduced cost, because it belongs to T_s.

Let $k^*(j)$ be the number of arcs of the shortest path from s to j. Then $k^*(j) < n$, because the shortest path contains at most $n - 1$ arcs.

Since the path is optimal $\overline{d}^{k^*(j)}(j) = 0$.
In general, for $k \neq k^*(j)$ we have $\overline{d}^k(j) \geq 0$.

Consider the directed walk of n arcs from s to j. Its reduced cost is non-negative: $\overline{d}^n(j) \geq 0$.

Hence $\overline{d}^n(j) - \overline{d}^{k^*(j)}(j) \geq 0$ and hence

$$\max_{0 \leq k \leq n-1} \{\overline{d}^n(j) - \overline{d}^k(j)\} \geq 0 \;\; \forall j \in \mathcal{N}.$$

Proof: $\mu^* = 0$ (5)

Lemma. There exists a node $p \in \mathcal{N}$ such that

$$\max_{0 \leq k \leq n-1} \{\overline{d}^n(p) - \overline{d}^k(p)\} = 0.$$

Proof. Let j be a node in the optimal cycle C.
Consider a walk of n arcs made by

- the shortest path from s to j;
- the arcs of C, starting from j and running along the cycle until n arcs have been selected (possibly with repetitions).

Let p be the final node of the walk.
The reduced cost of the walk is 0, because all its arcs are in $T_s \cup C$.
The walk must contain at least one cycle, because it has n arcs.
Canceling the cycles from the walk, an (s, p) path is left, with zero reduced cost and $\hat{k} \leq n - 1$ arcs.
Hence $\overline{d}^n(p) = \overline{d}^{\hat{k}}(p) = 0$, while for $k \neq \hat{k}$ we have $\overline{d}^k(p) \geq 0$.
This proves the lemma.

Proof: $\mu^* = 0$ (6)

When $\mu^* = 0$ we have proven that:

- $d^n(j) - d^k(j) = \overline{d}^n(j) - \overline{d}^k(j) \quad \forall j \in \mathcal{N} \ \ \forall k = 0, \ldots, n$
- $\max_{0 \le k \le n-1}\{\overline{d}^n(j) - \overline{d}^k(j)\} \ge 0 \ \ \forall j \in \mathcal{N}$
- $\exists p \in \mathcal{N} : \max_{0 \le k \le n-1}\{\overline{d}^n(p) - \overline{d}^k(p)\} = 0$

Therefore this proves the statement:

$$\mu^* = 0 = \min_{j \in \mathcal{N}} \max_{0 \le k \le n-1}\{\frac{d^n(j) - d^k(j)}{n-k}\}.$$