# Minimum cost spanning $r$-arborescence

## Combinatorial Optimization

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# The problem

Problem data:

- a digraph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$,
- a node $r \in \mathcal{N}$,
- a cost function $c : \mathcal{A} \to \Re_+$.

**Problem (Minimum Cost Spanning $r$-Arborescence Problem).** Find a spanning $r$-arborescence of minimum cost.

A digraph $\mathcal{T} = (\mathcal{N}, \mathcal{A})$ is a spanning rooted arborescence ($r$-arborescence, for short) if and only if there is a unique directed path from its root node $r \in \mathcal{N}$ to all the other nodes in $\mathcal{N} \setminus \{r\}$ and no directed path from any node in $\mathcal{N} \setminus \{r\}$ to $r$.

# Counter-example

The algorithms for the MSTP do not work.

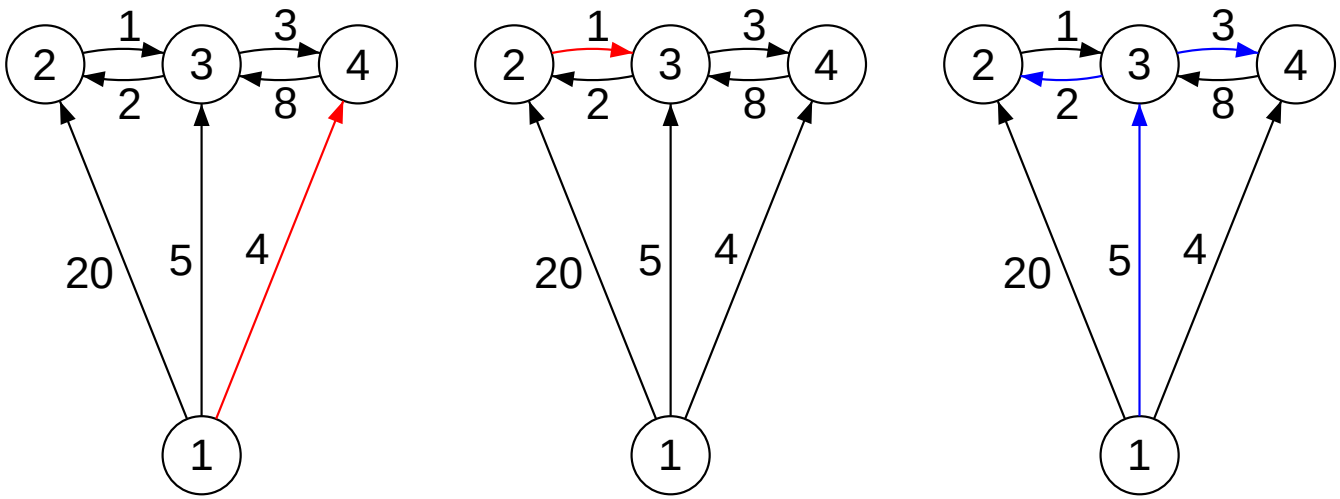



Figure: Prim algorithm would select $(1, 4)$ first. Kruskal algorithm would select $(2, 3)$ first. None of them belongs to the optimal solution.

# A mathematical programming model

$$\min z = \sum_{a \in \mathcal{A}} c_a x_a$$

$$\text{s.t.} \sum_{a \in \delta(S)} x_a \geq 1 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\} \qquad (1)$$

$$x_a \in \{0,1\} \qquad \forall a \in \mathcal{A}$$

A binary variable $x_a$ indicates whether each arc $a \in \mathcal{A}$ belongs to the solution or not.
We call $r$-cuts all arc subsets $\delta(S)$ corresponding to all node subsets $S$ not containing $r$, made of the arcs entering $S$:

$$\delta(S) = \{(i,j) \in \mathcal{A} : i \notin S \wedge j \in S\} \;\; \forall S \subset \mathcal{N} : r \notin S$$

Constraints (1) impose that all such subsets be reached by at least one entering arc.

Integrality conditions are redundant. We initially assume this and then we prove that the optimal solution obtained from the linear relaxation is integer.

# A mathematical programming model

The linear relaxation is

$$\min z = \sum_{a \in \mathcal{A}} c_a x_a$$

$$\text{s.t.} \sum_{a \in \delta(S)} x_a \geq 1 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

$$0 \leq x_a \leq 1 \qquad \forall a \in \mathcal{A}.$$

- All terms of the sums $\sum_{a \in \delta(S)} x_a$ are non-negative;
- the right-hand-side is equal to 1.

Therefore, no $x$ variable is required to take value larger than 1 to achieve feasibility.

The objective function coefficients $c$ are non-negative.

Therefore, no $x$ variable is required to take value larger than 1 to achieve optimality.

Therefore, the bounds $x_a \leq 1$ are redundant.

# The primal-dual pair

The primal problem.

$$\min z = \sum_{a \in \mathcal{A}} c_a x_a$$

$$\text{s.t.} \sum_{a \in \delta(S)} x_a \geq 1 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

$$x_a \geq 0 \qquad \forall a \in \mathcal{A}$$

The dual problem.

$$\max w = \sum_{S \subseteq \mathcal{N} \setminus \{r\}} y_S$$

$$\text{s.t.} \sum_{S \subseteq \mathcal{N} \setminus \{r\} : a \in \delta(S)} y_S \leq c_a \qquad \forall a \in \mathcal{A}$$

$$y_S \geq 0 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

# Complementary slackness conditions

Primal constraints and dual variables correspond to subsets $S$ non including $r$, i.e. to $r$-cuts.

Primal C.S.C.: $x_a(c_a - \sum_{S \subseteq \mathcal{N} \setminus \{r\} : a \in \delta(S)} y_S) = 0 \quad \forall a \in \mathcal{A}$

Dual C.S.C.: $y_S(\sum_{a \in \delta(S)} x_a - 1) = 0 \quad \forall S \subseteq \mathcal{N} \setminus \{r\}$

The initial primal solution is $x_a = 0 \; \forall a \in \mathcal{A}$ and it is primal infeasible (and super-optimal).

The initial dual solution is $y_S = 0 \; \forall S \subseteq \mathcal{N} \setminus \{r\}$ and it is dual feasible (and sub-optimal).

# Complementary slackness conditions

Owing to the constraints

$$\sum_{a \in \delta(S)} x_a \geq 1 \quad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

primal infeasibility is measured by the number of unreached node subsets (SCCs).

Owing to the primal C.S.C.

$$x_a (c_a - \sum_{S \subseteq \mathcal{N} \setminus \{r\} : a \in \delta(S)} y_S) = 0 \quad \forall a \in \mathcal{A}$$

the only arcs that can be used to traverse $r$-cuts are those with null reduced cost (admissible arcs):

$$x_a = 1 \Rightarrow \overline{c}_a = 0$$

where $\overline{c}_a = c_a - \sum_{S \subseteq \mathcal{N} \setminus \{r\} : a \in \delta(S)} y_S$.

# Edmonds algorithm

This algorithm is due to Chu and Liu (1965), Edmonds (1967), Bock (1971). It is a primal-dual algorithm. Iteratively it does the following:

- Dual iteration: select a violated primal constraint. It corresponds to an $r$-cut that does not contain any admissible arc: its correspondent SCC $S$ is not reached from $r$ through admissible arcs. Make the corresponding dual variable $y_S$ basic: increased it (dual ascent procedure) until it activates one or more dual constraints corresponding to one or more primal variables $x_a$ (arcs). Such arcs get zero reduced cost and therefore they become admissible.
- Primal iteration: select a primal variable $x_a$ corresponding to an admissible arc entering $S$ and make it basic: increase it to repair the infeasibility of the selected primal constraint. The $r$-cut corresponding to $S$ is now traversed by the selected basic arc $a$.

# Two approaches

In the scientific and educational literature Edmonds algorithm is presented in two different ways.

- **Description 1**:
  - no reference to duality theory, no dual data-structure;
  - the a priori knowledge of $r$ is assumed;
  - SCCs not reachable from $r$ are identified by a sequence of calls to a graph search sub-routine (such as DFS or BFS);
  - complexity: $O(mn)$.
- **Description 2**:
  - Step 1: build a dual data-structure;
  - Step 2: find the optimal solution;
  - the construction of the dual data-structure is independent of $r$;
  - better computational complexity, such as $O(m \log n)$, depending on the data-structure used;
  - very sophisticated implementations are possible.

# Edmonds algorithm: correctness (1)

Dual iterations.

Dual feasibility requires $\sum_{S \subseteq N \setminus \{r\}: a \in \delta(S)} y_S \leq c_a \quad \forall a \in A$. Therefore, when a dual variable $y_S$ enters the basis it takes a value equal to the minimum reduced cost among those of the arcs in the $r$-cut $\delta(S)$.
In this way dual feasibility is always kept.

Primal iterations.

Primal feasibility requires $\sum_{a \in \delta(S)} x_a \geq 1 \quad \forall S \subseteq \mathcal{N} \setminus \{r\}$. Therefore, when a primal variable $x_a$ enters the basis, it takes value 1, which is the minimum amount needed to satisfy at least one more primal constraint.
In this way primal feasibility is iteratively achieved.

Since the C.S.C. are satisfied at each iteration, when primal feasibility is achieved the dual solution is optimal.

# Edmonds algorithm: correctness (2)

To find a violated primal constraint, we need to find a SCC $S$, not including $r$, that is not reachable from $r$ through admissible arcs.

If some admissible arcs form a SCC, then reaching any node in it makes all the others reachable.

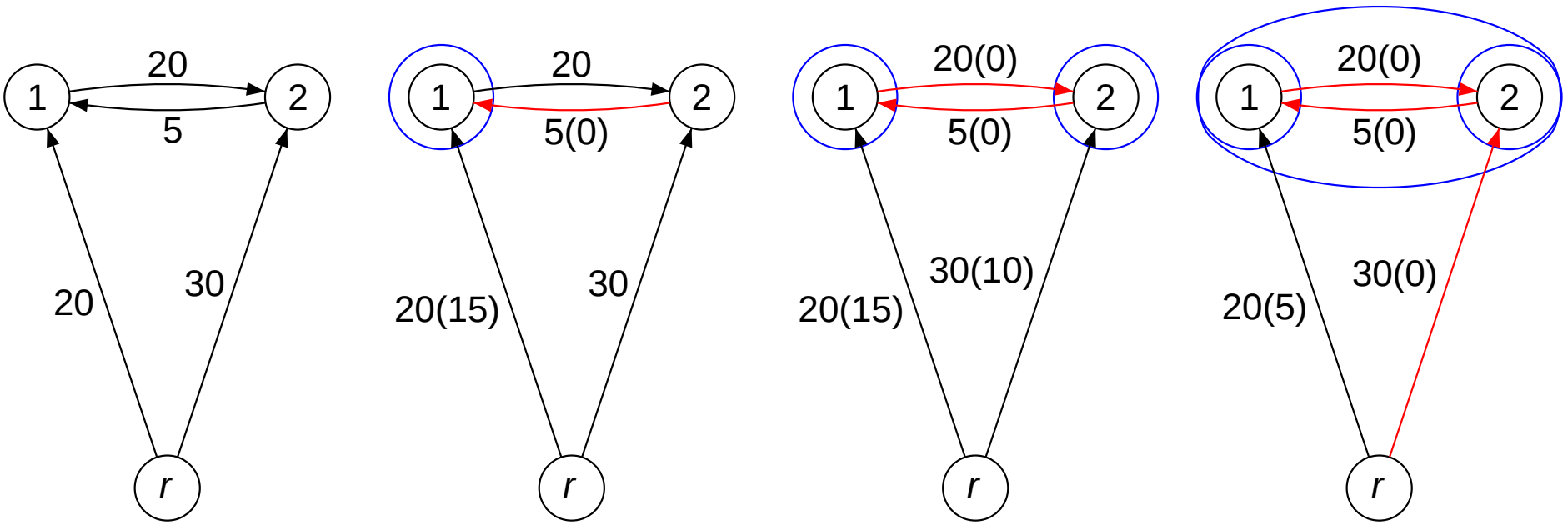
Therefore we consider the subgraph $\mathcal{D}_B = \{\mathcal{N}, \mathcal{B}\}$, where $\mathcal{B}$ is the set of admissible arcs.

If $\mathcal{D}_B$ contains a spanning $r$-arborescence $\mathcal{T}$, then $\mathcal{T}$ is a minimum cost spanning $r$-arborescence and the algorithm stops. Otherwise, there is a SCC $S$ in $\mathcal{D}_B$ such that

- $r \notin S$
- $\overline{c}_a > 0 \quad \forall a \in \delta(S)$.

Any SCC satisfying these properties can be chosen for the next iteration.

# Example




To retrieve a corresponding optimal primal solution we need to select some of the admissible arcs as basic (some others can be redundant).
In this example there are three admissible arcs: arcs $(r, 2)$ and $(2, 1)$ are basic; arc $(1, 2)$ is redundant.

# Edmonds algorithm: correctness (3)

Redundant arcs are identified by possibly removing them following the reverse insertion order: the latest inserted arc survives among those with the same head.

After removing redundant arcs, no two arcs entering the same node $v \in \mathcal{N} \setminus \{r\}$ can remain and all cycles (SCCs) are broken.

All basic variables have value 1.

Therefore the final primal solution is made of $n - 1$ arcs and therefore it is an $r$-arborescence.

# Edmonds algorithm: complexity (1)

Edmonds algorithm can be implemented with complexity $O(nm)$, because it requires at most $2(n-1)$ iterations and each of them has complexity $O(m)$.

Let $h$ be the number of SCCs of $\mathcal{D}_B$ (excluding $\{r\}$).
Let $h_0$ be the number of SCCs of $\mathcal{D}_B$ with no admissible entering arcs (excluding $\{r\}$).

At each iteration, the sum $h + h_0$ decreases by at least 1:

- if the selected SCC $S$ remains a SCC, then it gets an entering admissible arc: therefore, $h$ does not change and $h_0$ decreases;
- if $S$ is merged with another SCC, then $h$ decreases and $h_0$ does not increase.

Initially $h = n - 1$ and $h_0 = n - 1$. Eventually $h = h_0 = 0$. Therefore the iterations are at most $2(n-1)$.

# Edmonds algorithm: complexity (2)

In $O(m)$ time it is possible to find a SCC $S$ in $\mathcal{D}_B$, sorting the nodes in pre-topological order, starting from $r$, so that the first node belongs to a SCC not reachable from $r$ and with no admissible entering arcs.

Therefore each iteration has time complexity $O(m)$.

In this implementation

- the root $r$ must be known since the beginning;
- more than one arc may become admissible in a same iteration;
- no dual data-structure is required;
- any graph search algorithm can be used as a sub-routine (DFS, BFS,...).

# Iterative dual ascent

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**for** $a \in \mathcal{A}$ **do**
  $\overline{c}_a \leftarrow c_a$
*VisitGraphFw*
**while** $Order[n] \neq r$ **do**
  *VisitGraphBw*$(Order[n])$
  *FindArc*
  *UpdateCost*
  *VisitGraphFw*

---

Upon termination, a vector of predecessors, *Pred*, identifies the primal optimal solution.

# Procedure *VisitGraphFw*

**for** $i \in \mathcal{N}$ **do**
  $Pred[i] \leftarrow nil$
$Pred[0] \leftarrow 0$
$k \leftarrow 0$
DFS(0)

Node 0 is a dummy node connected to all others with null cost arcs $(0, i)\ \forall i \in \mathcal{N}$.
The vector *Pred* indicates the predecessors when the digraph $\mathcal{D}_B$ is visited.
The vector *Order* indicates the reverse pre-topological order of the nodes.
$k$ is the index of *Order*.

# Procedure DFS($i$)

---

**for** $j \in \delta^+(i)$ **do**
  **if** $(\overline{c}_{ij} = 0) \wedge (Pred[j] = nil)$ **then**
    $Pred[j] \leftarrow i$
    DFS($j$)
$k \leftarrow k + 1$
$Order[k] \leftarrow i$

---

DFS is a recursive procedure.

# Procedure *VisitGraphBw*

**for** $i \in \mathcal{N}$ **do**
  $Succ[i] \leftarrow nil$
$Succ[Order[n]] \leftarrow Order[n]$
$h \leftarrow 0$
RevDFS($Order[n]$)

The vector *Succ* indicates the successors when the graph is visited backward.
*S* is the characteristic vector of a strongly connected component.
*h* is its index.

# Procedure RevDFS($i$)

---

**for** $j \in \delta^-(i)$ **do**
  **if** $(\overline{c}_{ji} = 0) \wedge (Succ[j] = nil)$ **then**
    $Succ[j] \leftarrow i$
    RevDFS($j$)
$h \leftarrow h + 1$
$S[h] \leftarrow i$

---

RevDFS is a recursive procedure, idenntical to DFS, but with the reversed arcs.

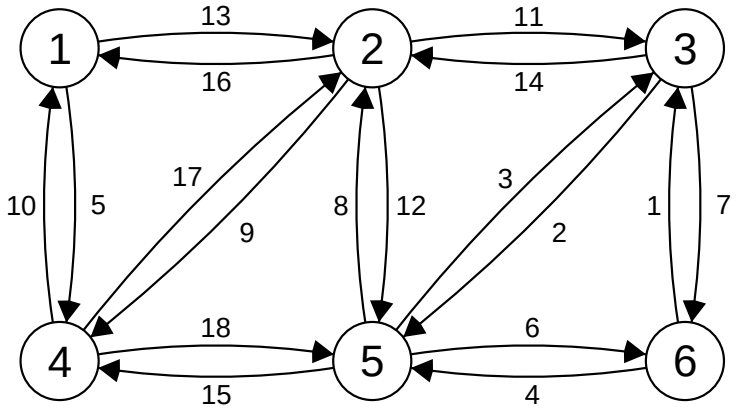
# Procedure *FindArc*

/* Find a minimum reduced cost arc entering $S$ */
**for** $i \in \mathcal{N}$ **do**
  $Flag[i] \leftarrow false$
**for** $u = 1, \ldots, h$ **do**
  $Flag[S[u]] \leftarrow true$
$\alpha \leftarrow \infty$
**for** $u = 1, \ldots, h$ **do**
  **for** $(i, S[u]) \in \delta^-(S[u])$ **do**
    **if** $(notFlag[i]) \wedge (\overline{c}_{iS[u]} < \alpha)$ **then**
      $\alpha \leftarrow \overline{c}_{iS[u]}$

# Procedure *UpdateCost*

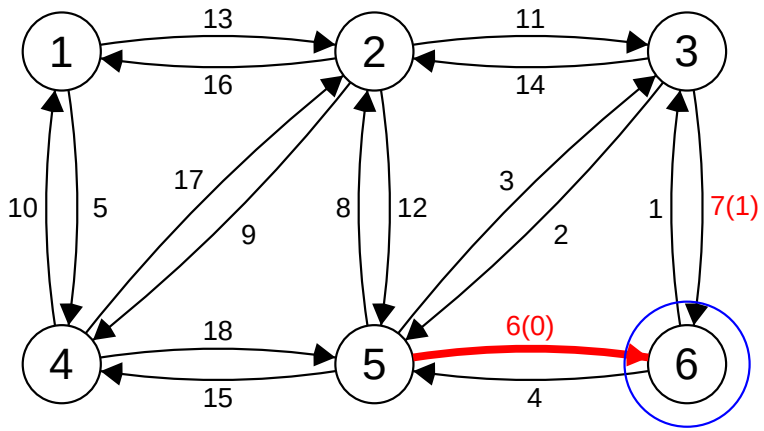
/* Update the reduced costs */
**for** $u = 1, \ldots, h$ **do**
  **for** $(i, S[u]) \in \delta^-(S[u])$ **do**
    **if** $(notFlag[i])$ **then**
      $\overline{c}_{iS[u]} \leftarrow \overline{c}_{iS[u]} - \alpha$

# Example




Node 1 is the root.

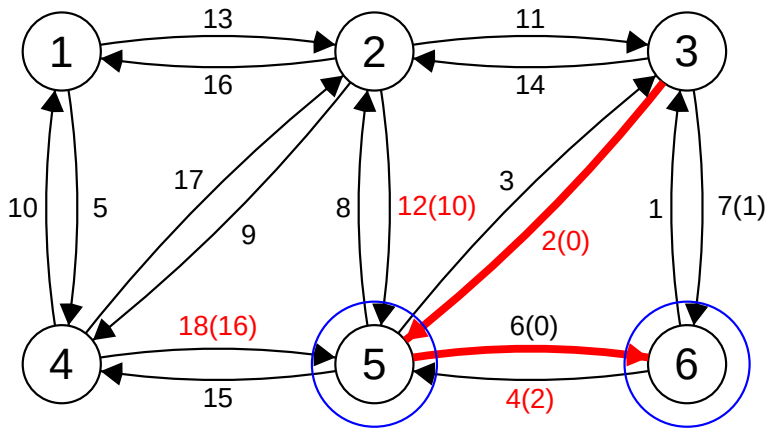
# Iteration 1




*DFS* : 1 2 3 4 5 6
*Order* : 1 2 3 4 5 6
$S = \{6\}$
$\delta^-(S) = \{(3,6),(5,6)\}$
$\alpha = 6$

$y_6 = 6$

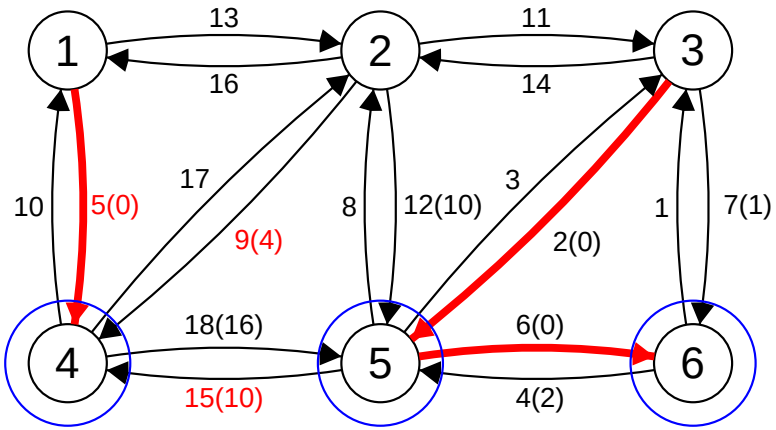
# Iteration 2




$DFS$ : 1 2 3 4 5 6
$Order$ : 1 2 3 4 6 5
$S = \{5\}$
$\delta^-(S) = \{(2,5),(3,5),(4,5),(6,5)\}$
$\alpha = 2$

$y_6 = 6$ $y_5 = 2$

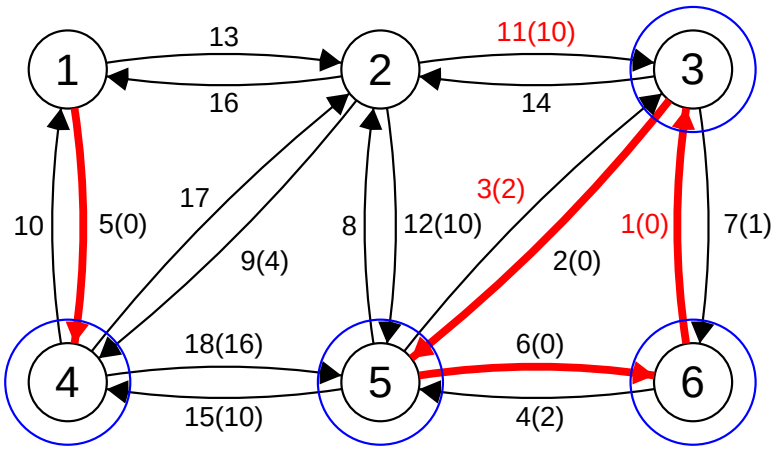
# Iteration 3




*DFS* : 1 2 3 5 6 4
*Order* : 1 2 6 5 3 4
$S = \{4\}$
$\delta^-(S) = \{(1,4),(2,4),(5,4)\}$
$\alpha = 5$

$y_6 = 6\ y_5 = 2\ y_4 = 5$

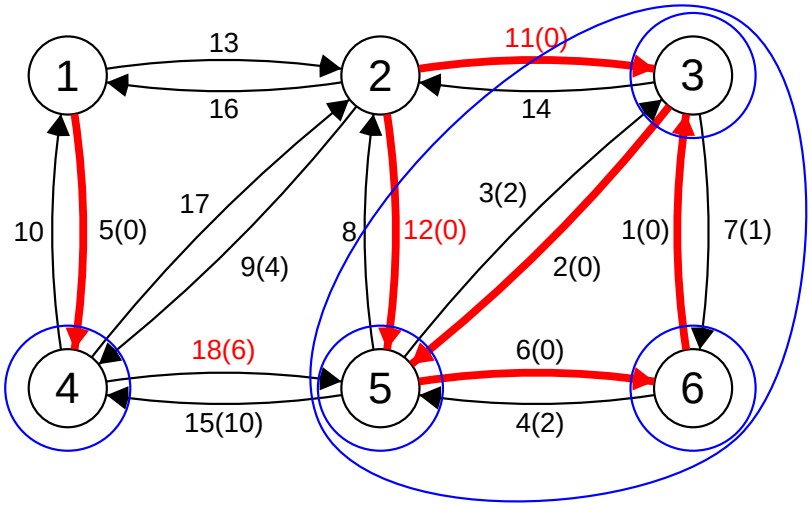
# Iteration 4




*DFS* : 1 4 2 3 5 6
*Order* : 4 1 2 6 5 3
$S = \{3\}$
$\delta^-(S) = \{(2,3),(5,3),(6,3)\}$
$\alpha = 1$

$y_6 = 6\ y_5 = 2\ y_4 = 5\ y_3 = 1$

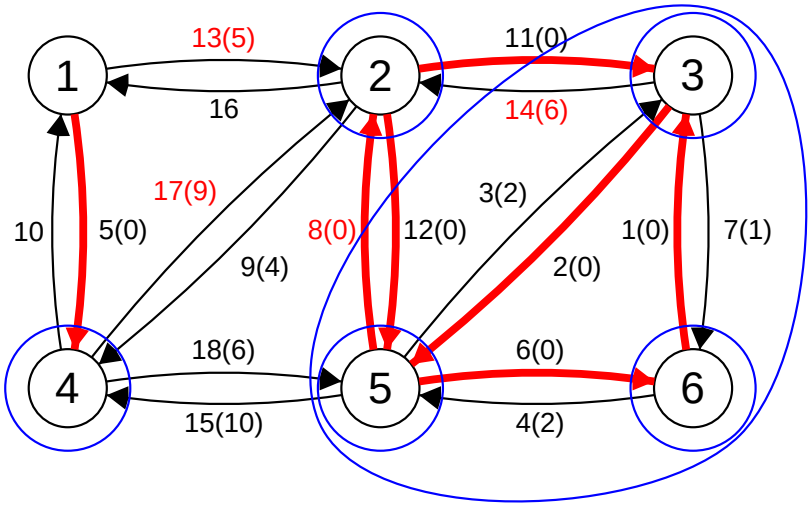
# Iteration 5




$DFS$ : 1 4 2 3 5 6
$Order$ : 4 1 2 6 5 3
$S = \{3, 5, 6\}$
$\delta^-(S) = \{(2, 3), (2, 5), (4, 5)\}$
$\alpha = 10$

$y_6 = 6$ $y_5 = 2$ $y_4 = 5$ $y_3 = 1$ $y_{356} = 10$

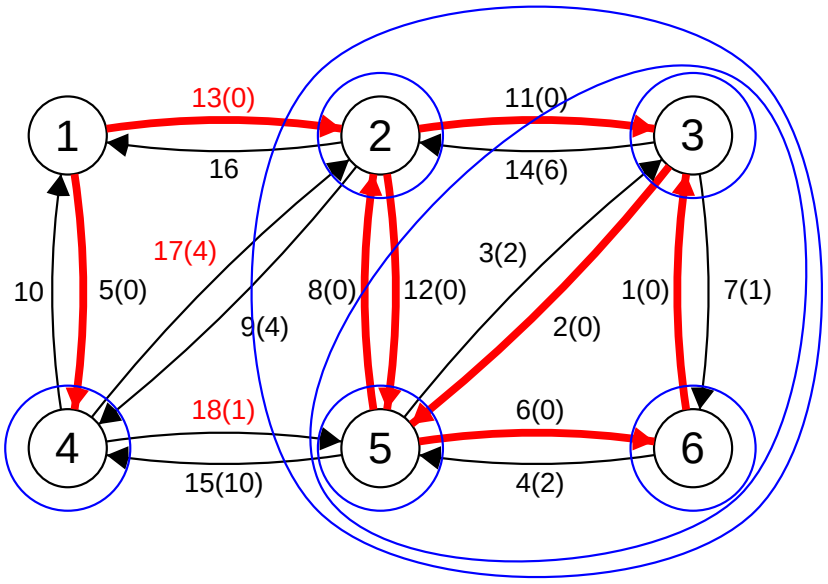
# Iteration 6




*DFS* : 1 4 2 3 5 6
*Order* : 4 1 6 5 3 2
$S = \{2\}$
$\delta^-(S) = \{(1,2),(3,2),(4,2),(5,2)\}$
$\alpha = 8$

$y_6 = 6$ $y_5 = 2$ $y_4 = 5$ $y_3 = 1$ $y_{356} = 10$ $y_2 = 8$

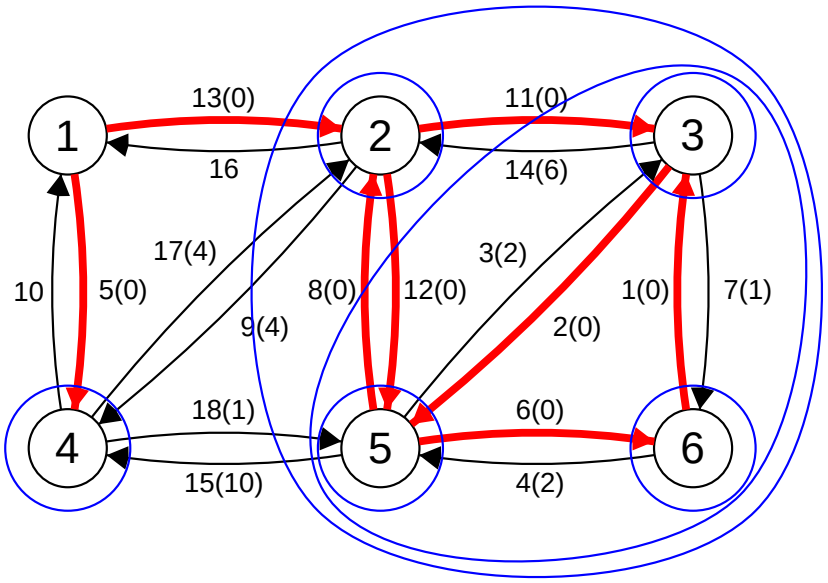
# Iteration 7




*DFS* : 1 4 2 3 5 6
*Order* : 4 1 6 5 3 2
$S = \{2, 3, 5, 6\}$
$\delta^-(S) = \{(1, 2), (4, 2), (4, 5)\}$
$\alpha = 5$

$y_6 = 6$ $y_5 = 2$ $y_4 = 5$ $y_3 = 1$ $y_{356} = 10$ $y_2 = 8$ $y_{2356} = 5$

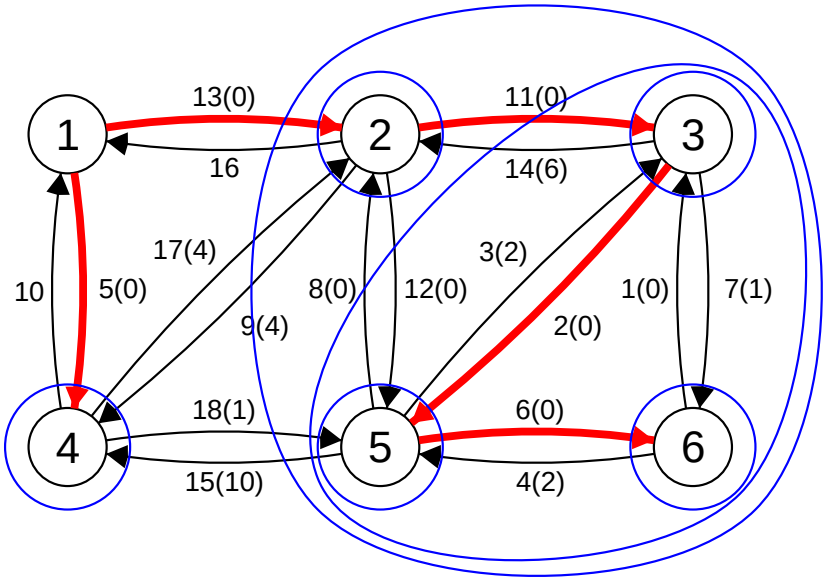
# Iteration 8




*DFS* : 1 4 2 3 5 6
*Order* : 4 6 5 3 2 **1**
Node 1 is the root: STOP!

$y_6 = 6\ y_5 = 2\ y_4 = 5\ y_3 = 1\ y_{356} = 10\ y_2 = 8\ y_{2356} = 5$

# Optimal solution




$$y_6 = 6\ y_5 = 2\ y_4 = 5\ y_3 = 1\ y_{356} = 10\ y_2 = 8\ y_{2356} = 5$$

$$w = \sum_S y_S = \mathbf{37}$$

# Implementation 2

In this second approach, Edmonds algorithm is implemented using both primal and dual data-structures.
The implementation does not make use of graph search sub-routines and assumes that the digraph is strongly connected.

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*Step*1 : *DualAscent*
*Step*2 : *ComputeSolution*$(r)$

---

- Step 1: build a sequence of nested SCCs in the digraph of admissible arcs, until a unique SCC is found containing all the nodes. The information about which node is the root is ignored. This step can be implemented in $O(m \log n)$.
- Step 2: find an optimal solution given a root $r$ and the information stored in the dual data-structure built in Step 1.
  This step can be implemented in $O(n)$.

# Step 1: dual ascent

SCCs with no basic entering arcs are iteratively found, so that they form a path. Each node in the path is a basic SCC.

A minimum cost arc $(u, v) \in \mathcal{A}$ is found among those entering the first node in the path, i.e. a certain SCC $S$; the dual variable $y_S$ is updated accordingly (dual ascent).

If $u$ belongs to a SCC $S'$ already in the path, then all SCCs in the path between $S$ and $S'$ are merged into a single SCC.
Otherwise, a new SCC including only node $u$ is created and appended at the beginning of the path.

Besides concatenating basic SCCs in a path, the algorithm also records how the basic SCCs are nested in one another forming a tree of basic SCCs.

# Step 1: data structures

For every SCC $S$ made of a single node, we define a priority queue $P(S)$ with the arcs entering it. When two SCCs $S'$ and $S''$ are merged into a larger one, $S$, the corresponding priority queues $P(S')$ and $P(S'')$ are merged into a single priority queue $P(S)$.

A union-find data-structure *Comp* is used to find the SCCs to which specific nodes belong.

Data structures:
A vector *Arc*: the (unique) basic entering arc for each basic SCC.
A vector *y*: the values of the basic dual variables.
A vector *Next*: the next SCC for each SCC in the path.
A vector *Parent*: the parent SCC for each SCC in the tree.
A list *Children*: information symmetric to that of *Parent*.

All these vectors have maximum size $2n$.

# Step 1: pseudo-code

*InitializeStars*
$k \leftarrow 0$
*NewComponent*(*Random*($\mathcal{N}$))
$s \leftarrow 1$
**while** $P(s) \neq \emptyset$ **do**
  **repeat**
    $(u, v) \leftarrow$ *ExtractMin*($P(s)$)
  **until** $Comp[u] \neq s$
  $Arc[s] \leftarrow (u, v)$
  $y[s] \leftarrow c(u, v)$
  *ReduceCosts*($P(s), y[s]$)
  **if** $Comp[u] = nil$ **then**
    *NewComponent*($u$)
    $Next[k] \leftarrow s$
  **else**
    *Merge*($Comp[u]$)
  $s \leftarrow k$

# InitializeStars

**for** $i \in \mathcal{N}$ **do**
  $P(i) \leftarrow nil$
  $Comp[i] \leftarrow nil$
**for** $(u, v) \in \mathcal{A}$ **do**
  $InsertStar(u, c(u, v), \delta(v))$

# CreateNewSCC

$k \leftarrow k + 1$
$Arc[k] \leftarrow nil$
$y[k] \leftarrow 0$
$Parent[k] \leftarrow nil$
$Children[k] \leftarrow nil$

# NewComponent(u)

*CreateNewSCC*
$Comp[u] \leftarrow k$
$Leaf[u] \leftarrow k$
$P[k] \leftarrow \delta[u]$

# Merge(t)

*CreateNewSCC*
$P[k] \leftarrow nil$
$Next[k] \leftarrow Next[t]$
$Stop \leftarrow false$
**repeat**
  $Parent[s] \leftarrow k$
  $Insert(s, Children(k))$
  $P[k] \leftarrow MergePQ(P[s], P[k])$
  $UpdateComp(s, k)$
  **if** $s = t$ **then**
    $Stop \leftarrow true$
  **else**
    $s \leftarrow Next[s]$
**until** *Stop*

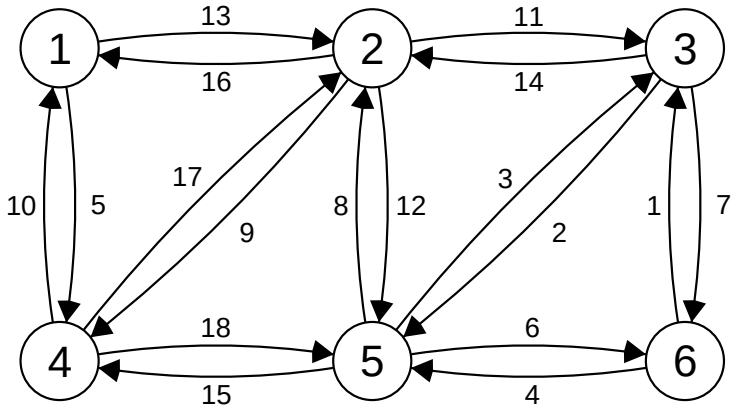
# Computational complexity of Step 1

The number of calls to *InsertPQ* is $O(m)$ (all arcs are inserted in one PQ).
The number of calls to *ExtractMin* is $O(m)$ (each arc can be extracted at most once).
The number of calls to *MergePQ* is $O(n)$ (at most $2n$ SCCs become basic).
The number of calls to *ReduceCosts* is $O(n)$ (see above).

If all operations are implemented to take $O(\log n)$, then the overall complexity is $O(m \log n)$.

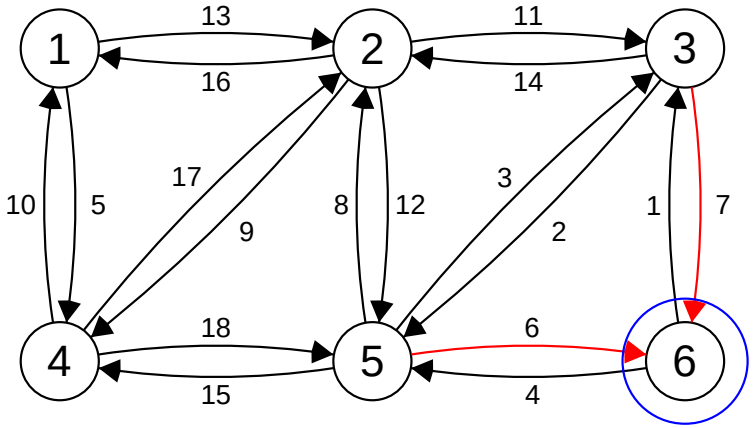
*UpdateComp* requires a Union-Find data-structure.

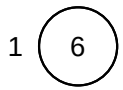
# Example of Step 1




Assume node 6 is selected at random to start.

# Dual iteration 1



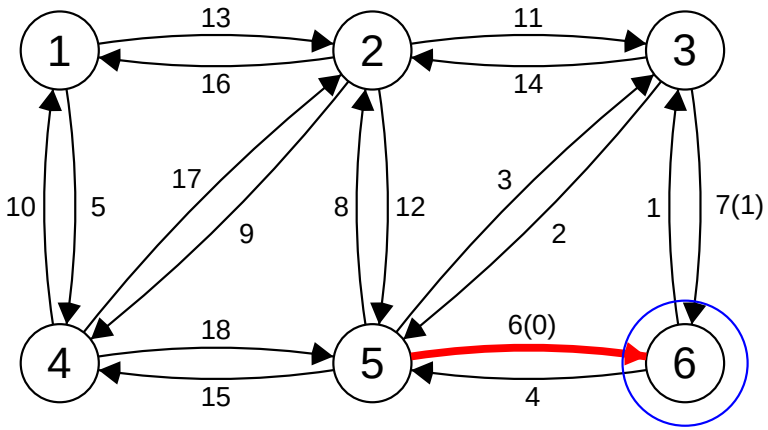

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | | | | | 1 |
| Leaf | | | | | | 1 |

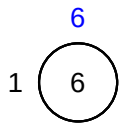
$k = 1 \quad s = 1$

$P(1) = \{((5, 6), 6), ((3, 6), 7)\}$

| $k$ | 1 |
|---|---|
| Arc | nil |
| y | 0 |
| Next | nil |
| Parent | nil |
| Children | nil |

# Primal iteration 1



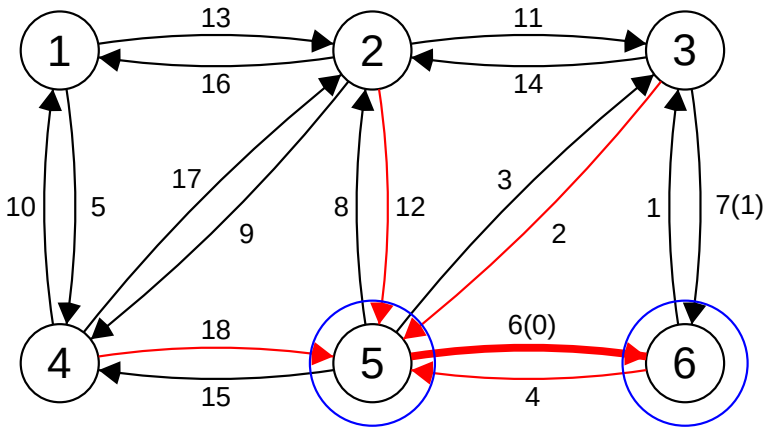



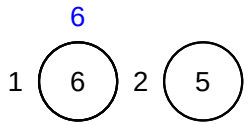

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | | | | | 1 |
| Leaf | | | | | | 1 |

$k = 1 \quad s = 1$
$(u, v) = (5, 6) \quad \overline{c}_{56} = 6$
$P(1) = \{((3, 6), 1)\}$

| $k$ | 1 |
|---|---|
| Arc | (5,6) |
| y | 6 |
| Next | nil |
| Parent | nil |
| Children | nil |

# Dual iteration 2




| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | | | | 2 | 1 |
| Leaf | | | | | 2 | 1 |

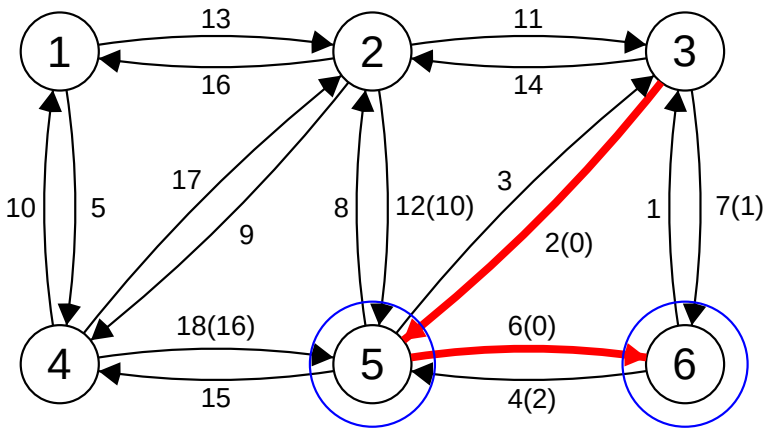
$k = 2 \quad s = 1$

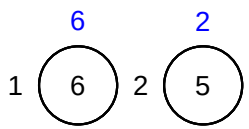
$P(2) =$

$\{((3,5),2),((6,5),4),((2,5),12),((4,5),18)\}$

| $k$ | 1 | 2 |
|---|---|---|
| Arc | (5,6) | nil |
| y | 6 | 0 |
| Next | nil | 1 |
| Parent | nil | nil |
| Children | nil | nil |

# Primal iteration 2



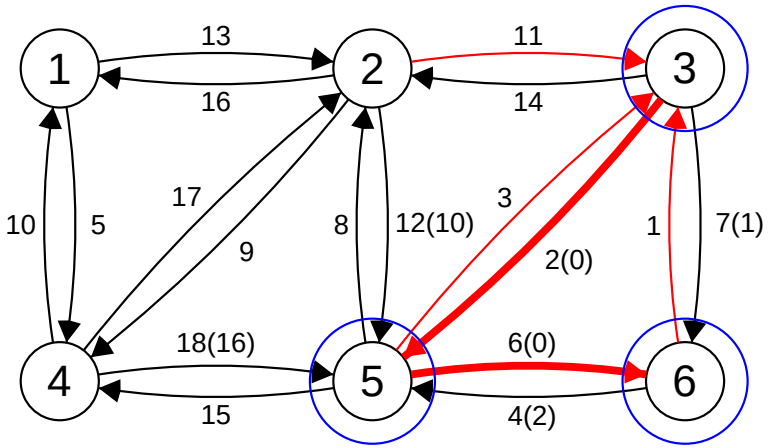



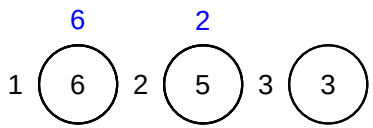

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | | | | 2 | 1 |
| Leaf | | | | | 2 | 1 |

$k = 2 \quad s = 2$
$(u, v) = (3, 5) \quad \overline{c}_{35} = 2$
$P(2) =$
$\{((6, 5), 2), ((2, 5), 10), ((4, 5), 16)\}$

| $k$ | 1 | 2 |
|---|---|---|
| Arc | (5,6) | (3,5) |
| y | 6 | 2 |
| Next | nil | 1 |
| Parent | nil | nil |
| Children | nil | nil |

# Dual iteration 3



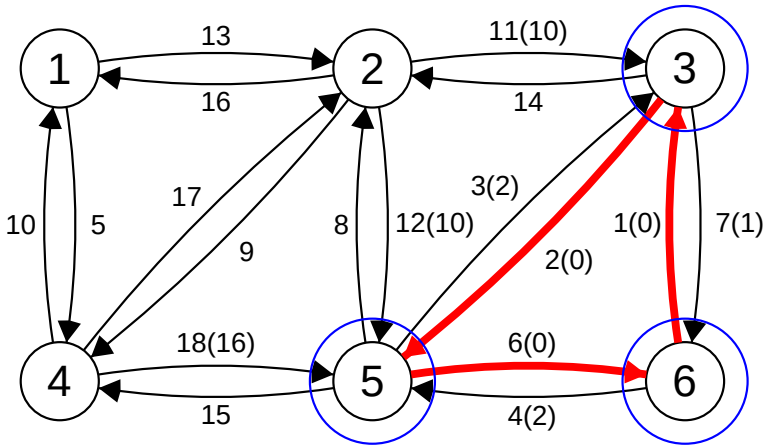

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | | 3 | | 2 | 1 |
| Leaf | | | 3 | | 2 | 1 |

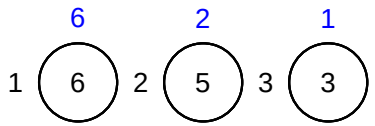
$k = 3 \quad s = 2$

$P(3) = \{((6, 3), 1), ((5, 3), 3), ((2, 3), 11)\}$

| $k$ | 1 | 2 | 3 |
|---|---|---|---|
| Arc | (5,6) | (3,5) | nil |
| y | 6 | 2 | 0 |
| Next | nil | 1 | nil |
| Parent | nil | nil | nil |
| Children | nil | nil | nil |

# Primal iteration 3



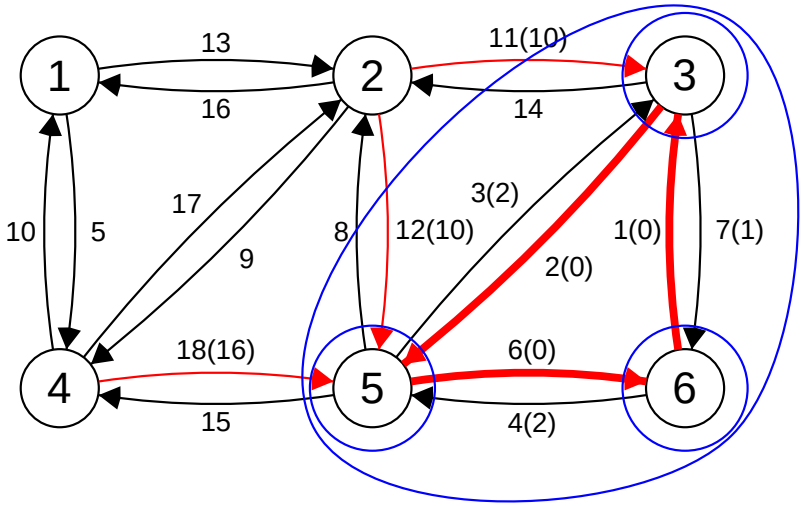



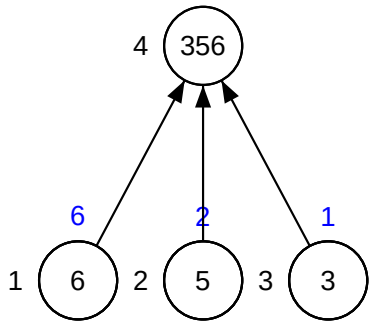

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | | 3 | | 2 | 1 |
| Leaf | | | 3 | | 2 | 1 |

$k = 3 \quad s = 3$
$(u, v) = (6, 3) \quad \overline{c}_{63} = 1$
$P(3) = \{((5, 3), 2), ((2, 3), 10)\}$

| $k$ | 1 | 2 | 3 |
|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) |
| y | 6 | 2 | 1 |
| Next | nil | 1 | nil |
| Parent | nil | nil | nil |
| Children | nil | nil | nil |

# Dual iteration 4



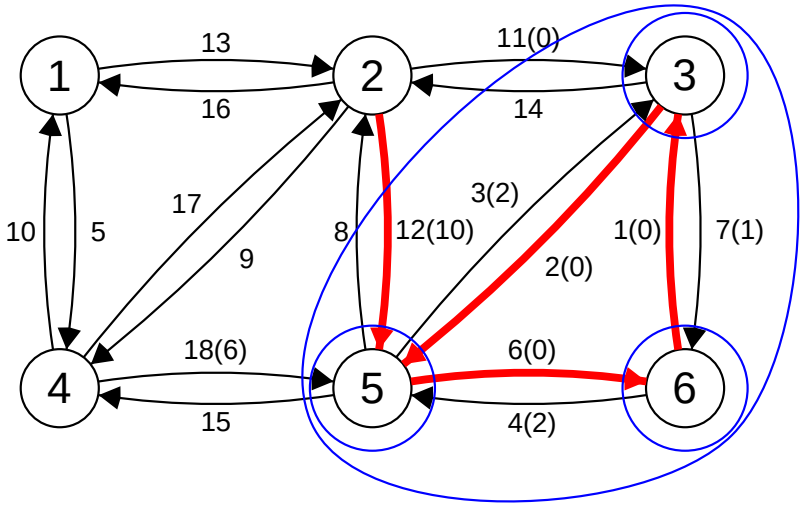



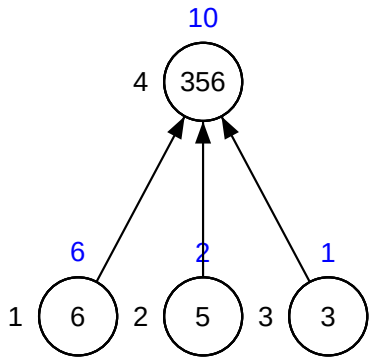

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | | 4 | | 4 | 4 |
| Leaf | | | 3 | | 2 | 1 |

$k = 4 \quad s = 3$
Merge $t = 1$
$P(4) = \{((3,6),1),((6,5),2),((5,3),2)$
$((2,5),10),((2,3),10),((4,5),16)\}$.

| $k$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | nil |
| y | 6 | 2 | 1 | 0 |
| Next | nil | 1 | nil | nil |
| Parent | 4 | 4 | 4 | nil |
| Children | nil | nil | nil | (1,2,3) |

# Primal iteration 4



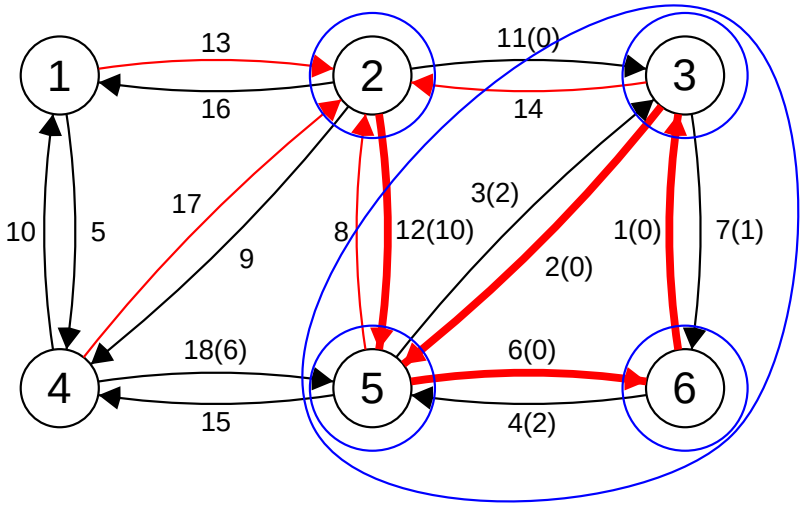



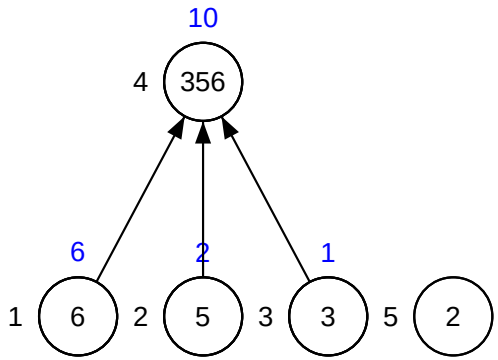

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | | 4 | | 4 | 4 |
| Leaf | | | 3 | | 2 | 1 |

$k = 4 \quad s = 4$
$(u, v) = (2, 5) \quad \overline{c}_{25} = 10$
$P(4) = \{((2, 3), 0), ((4, 5), 6)\}$

| $k$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) |
| y | 6 | 2 | 1 | 10 |
| Next | nil | 1 | nil | nil |
| Parent | 4 | 4 | 4 | nil |
| Children | nil | nil | nil | (1,2,3) |

# Dual iteration 5




| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | 5 | 4 | | 4 | 4 |
| Leaf | | 5 | 3 | | 2 | 1 |

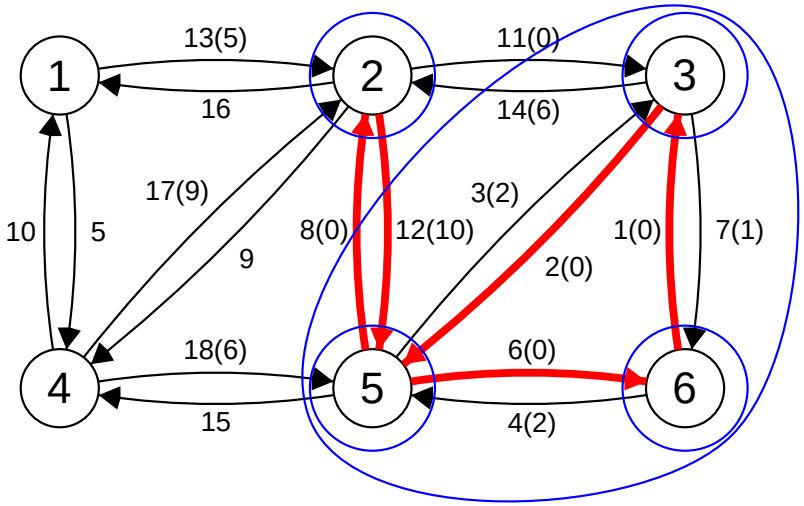
$k = 5 \quad s = 4$

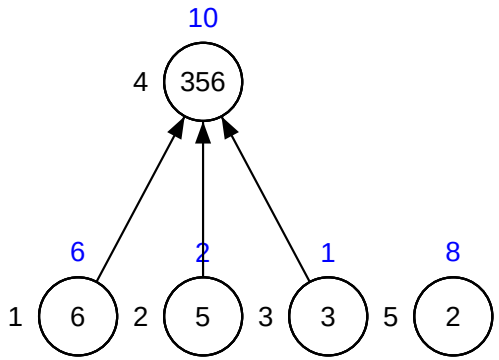
$P(5) =$

$\{((5,2),8),((1,2),13),((3,2),14),((4,2),17)\}.$

| $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | nil |
| y | 6 | 2 | 1 | 10 | 0 |
| Next | nil | 1 | nil | nil | 4 |
| Parent | 4 | 4 | 4 | nil | nil |
| Children | nil | nil | nil | (1,2,3) | nil |

# Primal iteration 5



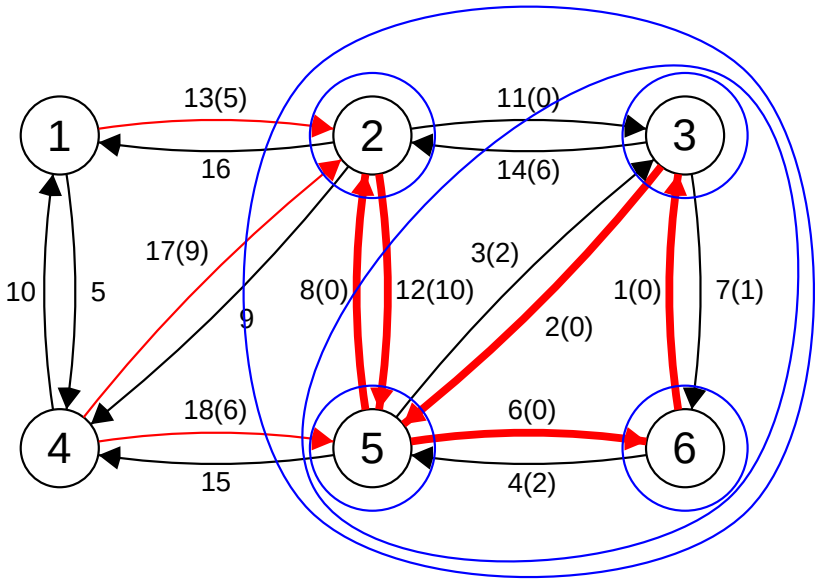



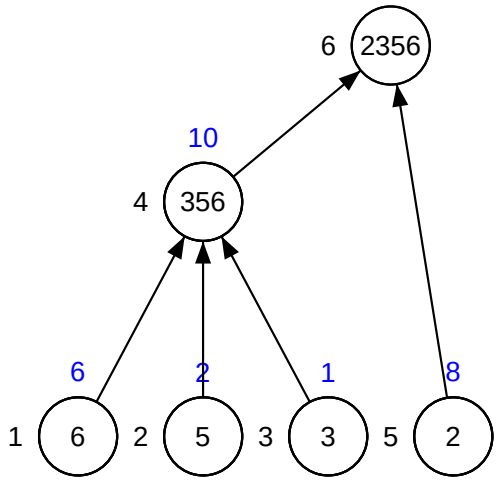

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | 5 | 4 | | 4 | 4 |
| Leaf | | 5 | 3 | | 2 | 1 |

$k = 5 \quad s = 5$
$(u, v) = (5, 2) \quad \overline{c}_{52} = 8$
$P(5) = \{((1, 2), 5), ((3, 2), 6), ((4, 2), 9)\}$

| $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | (5,2) |
| y | 6 | 2 | 1 | 10 | 8 |
| Next | nil | 1 | nil | nil | 4 |
| Parent | 4 | 4 | 4 | nil | nil |
| Children | nil | nil | nil | (1,2,3) | nil |

# Dual iteration 6




| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | 6 | 6 | | 6 | 6 |
| Leaf | | 5 | 3 | | 2 | 1 |

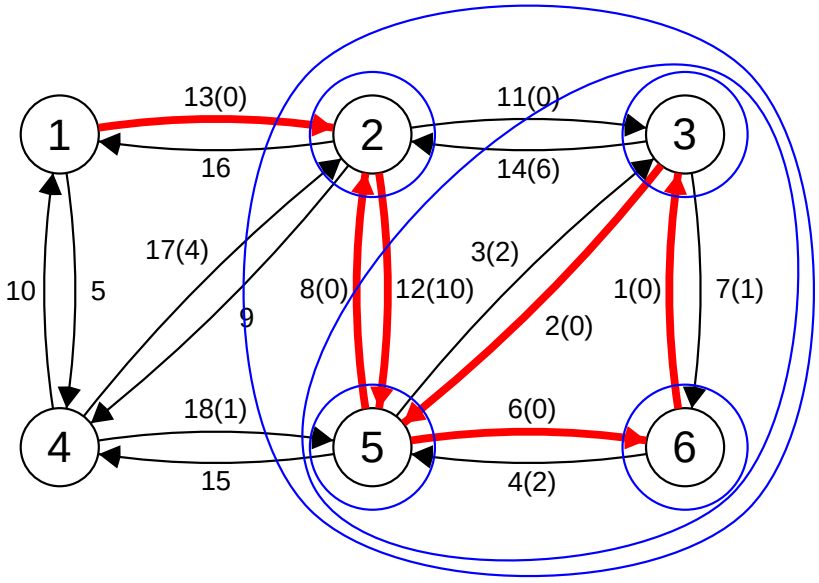
$k = 6 \quad s = 5$

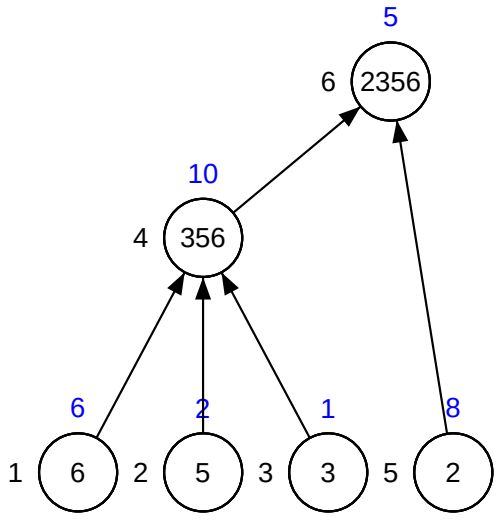
Merge $t = 4$.

$P(6) = \{((2,3),0),((1,2),5),((4,5),6)$
$((3,2),6),((4,2),9)\}$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | (5,2) | nil |
| y | 6 | 2 | 1 | 10 | 8 | 0 |
| Next | nil | 1 | nil | nil | 4 | nil |
| Parent | 4 | 4 | 4 | 6 | 6 | nil |
| Children | nil | nil | nil | (1,2,3) | nil | (4,5) |

# Primal iteration 6




| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | | 6 | 6 | | 6 | 6 |
| Leaf | | 5 | 3 | | 2 | 1 |

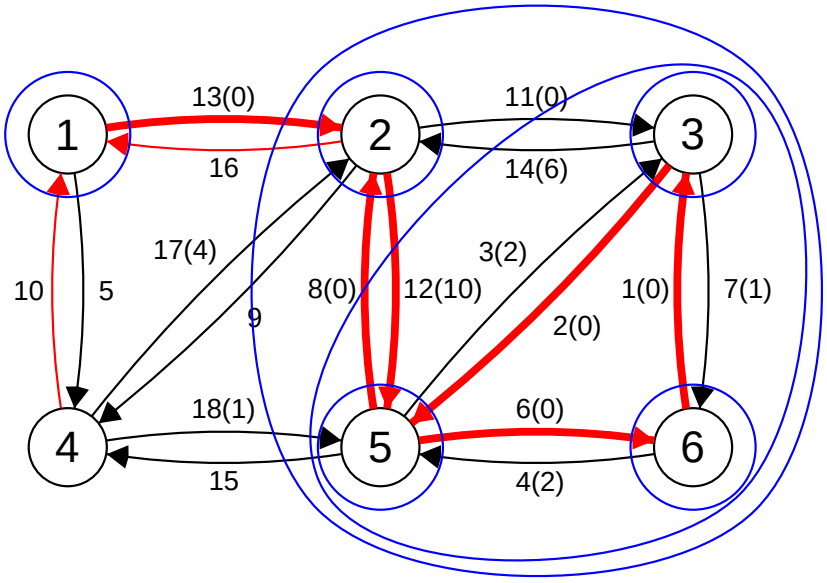
$k = 6 \quad s = 6$

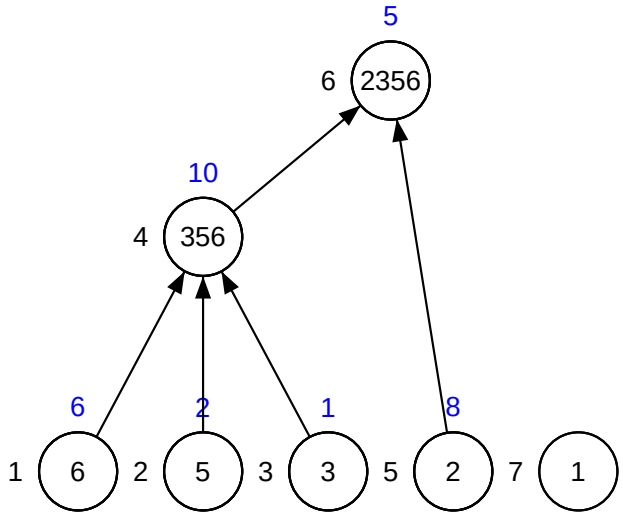
$(u, v) = (1, 2) \quad \overline{c}_{12} = 5$

$P(6) = \{((4, 5), 1), ((4, 2), 4)\}$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | (5,2) | (1,2) |
| y | 6 | 2 | 1 | 10 | 8 | 5 |
| Next | nil | 1 | nil | nil | 4 | nil |
| Parent | 4 | 4 | 4 | 6 | 6 | nil |
| Children | nil | nil | nil | (1,2,3) | nil | (4,5) |

# Dual iteration 7



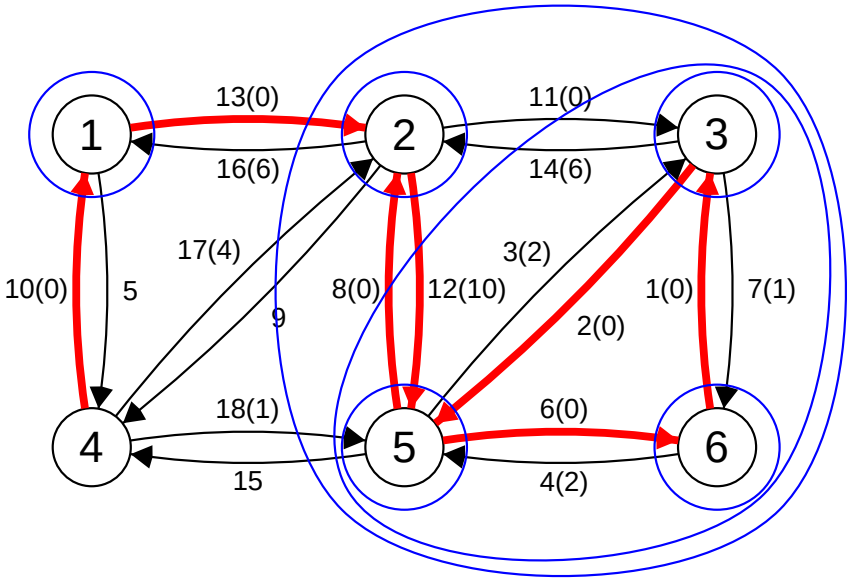

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | 7 | 6 | 6 | | 6 | 6 |
| Leaf | 7 | 5 | 3 | | 2 | 1 |

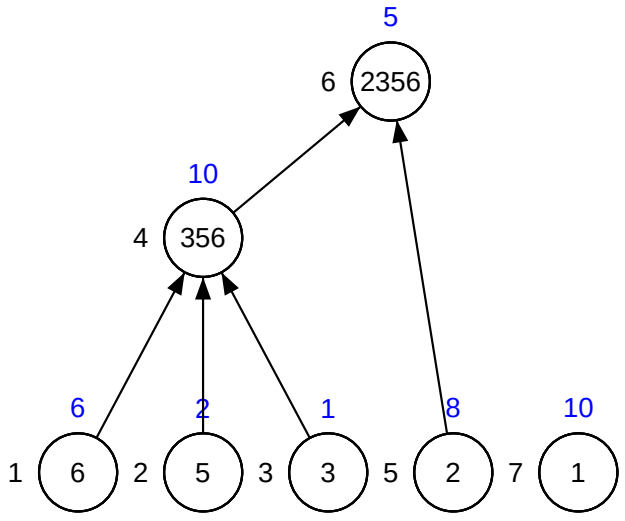
$k = 7 \quad s = 6$

$P(7) = \{((4,1),10),((2,1),16)\}$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | (5,2) | (1,2) | nil |
| y | 6 | 2 | 1 | 10 | 8 | 5 | 0 |
| Next | nil | 1 | nil | nil | 4 | nil | 6 |
| Parent | 4 | 4 | 4 | 6 | 6 | nil | nil |
| Children | nil | nil | nil | (1,2,3) | nil | (4,5) | nil |

# Primal iteration 7




| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | 7 | 6 | 6 | | 6 | 6 |
| Leaf | 7 | 5 | 3 | | 2 | 1 |

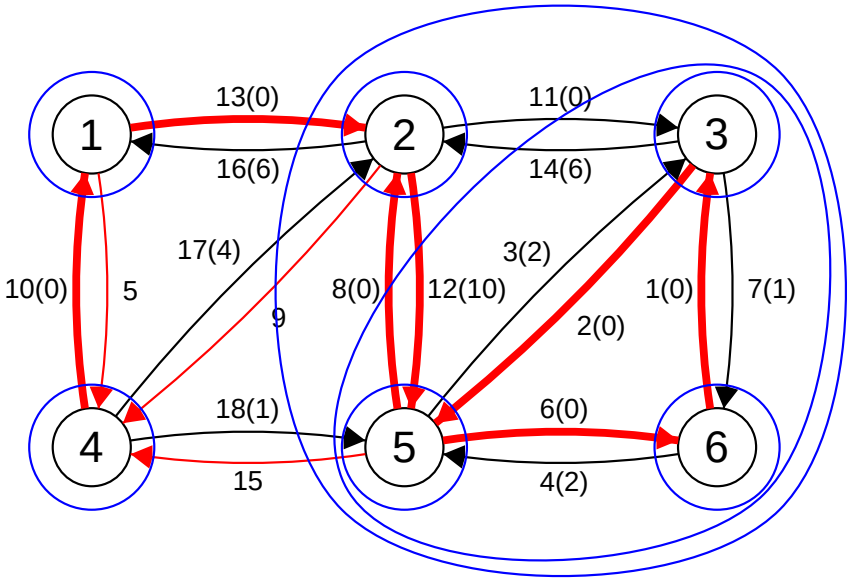
$k = 7 \quad s = 7$

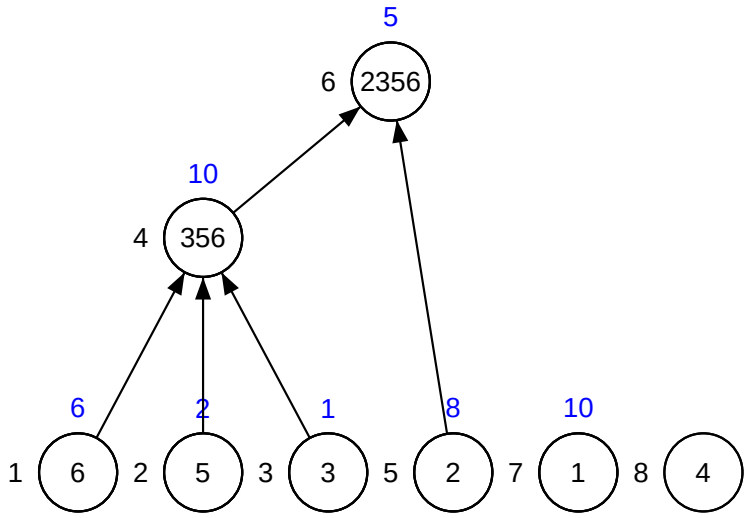
$(u, v) = (4, 1) \quad \overline{c}_{41} = 10$

$P(7) = \{((2, 1), 6)\}$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | (5,2) | (1,2) | (4,1) |
| y | 6 | 2 | 1 | 10 | 8 | 5 | 10 |
| Next | nil | 1 | nil | nil | 4 | nil | 6 |
| Parent | 4 | 4 | 4 | 6 | 6 | nil | nil |
| Children | nil | nil | nil | (1,2,3) | nil | (4,5) | nil |

# Dual iteration 8



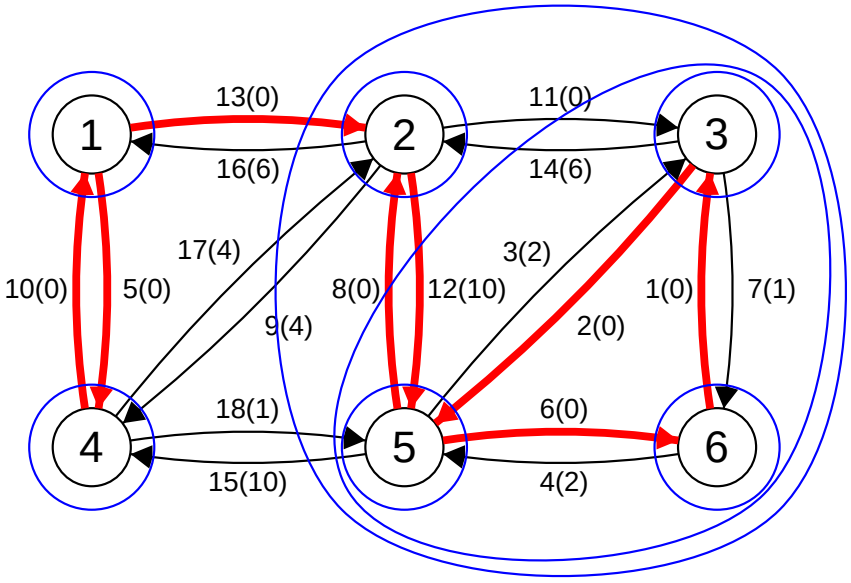

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | 7 | 6 | 6 | 8 | 6 | 6 |
| Leaf | 7 | 5 | 3 | 8 | 2 | 1 |

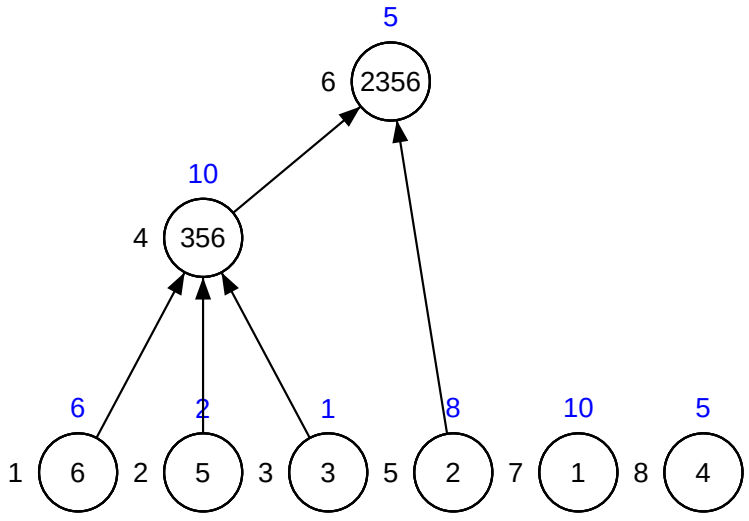
$k = 8 \quad s = 7$

$P(8) = \{((1,4),5),((2,4),9),((5,4),15)\}$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | (5,2) | (1,2) | (4,1) | nil |
| y | 6 | 2 | 1 | 10 | 8 | 5 | 10 | 0 |
| Next | nil | 1 | nil | nil | 4 | nil | 6 | 7 |
| Parent | 4 | 4 | 4 | 6 | 6 | nil | nil | nil |
| Children | nil | nil | nil | (1,2,3) | nil | (4,5) | nil | nil |

# Primal iteration 8




| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | 7 | 6 | 6 | 8 | 6 | 6 |
| Leaf | 7 | 5 | 3 | 8 | 2 | 1 |

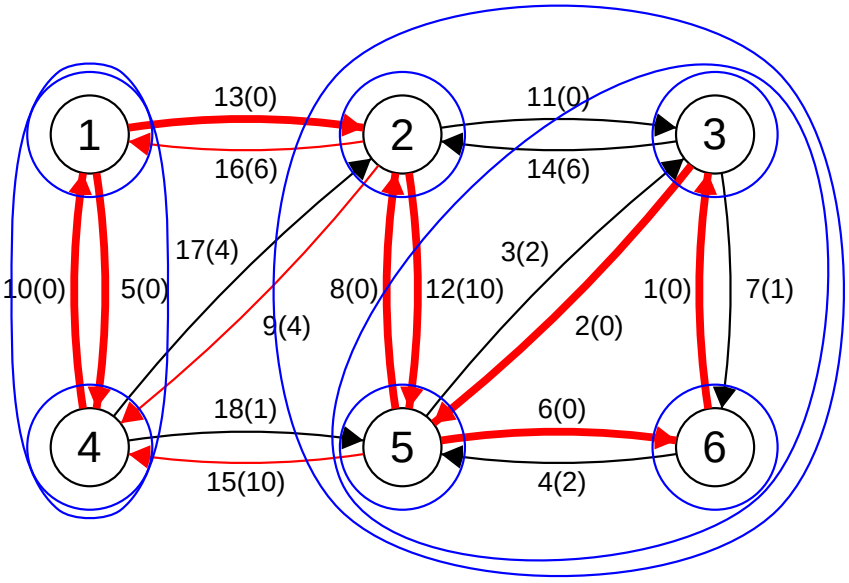
$k = 8 \quad s = 8$

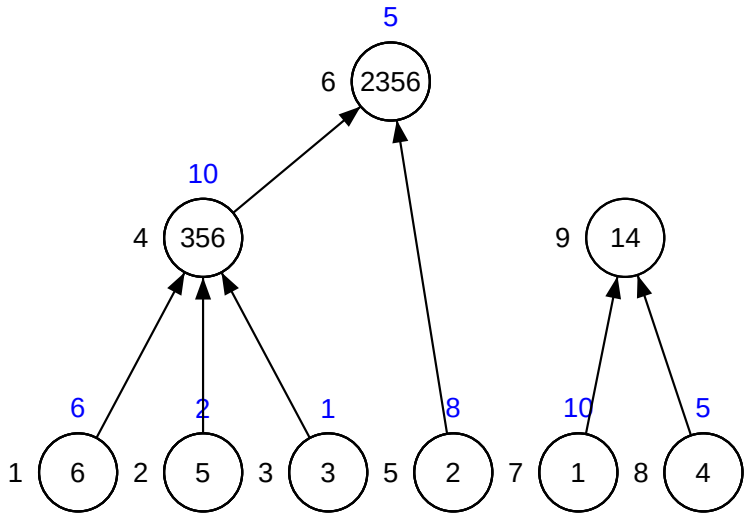
$(u, v) = (1, 4) \quad \overline{c}_{14} = 5$

$P(8) = \{((2, 4), 4), ((5, 4), 10)\}$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | (5,2) | (1,2) | (4,1) | (1,4) |
| y | 6 | 2 | 1 | 10 | 8 | 5 | 10 | 5 |
| Next | nil | 1 | nil | nil | 4 | nil | 6 | 7 |
| Parent | 4 | 4 | 4 | 6 | 6 | nil | nil | nil |
| Children | nil | nil | nil | (1,2,3) | nil | (4,5) | nil | nil |

# Dual iteration 9




| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | 9 | 6 | 6 | 9 | 6 | 6 |
| Leaf | 7 | 5 | 3 | 8 | 2 | 1 |

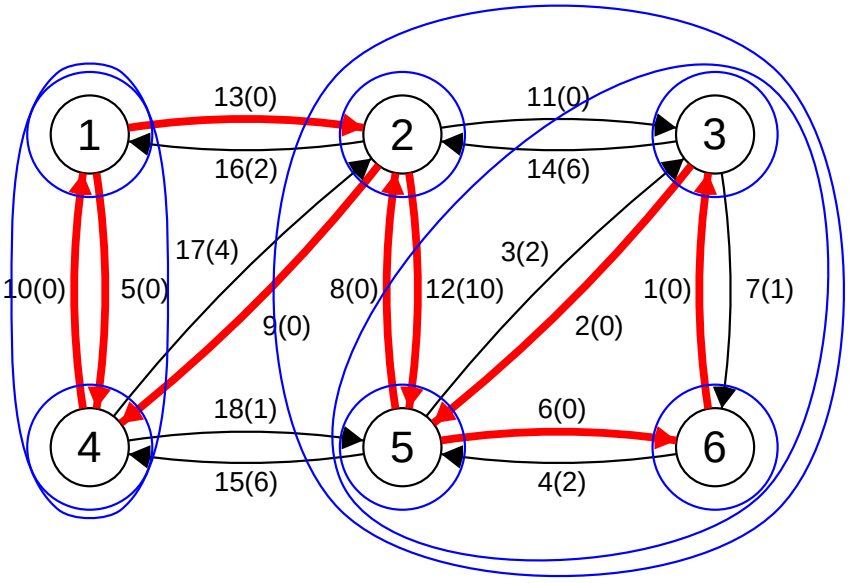
$k = 9 \quad s = 8$

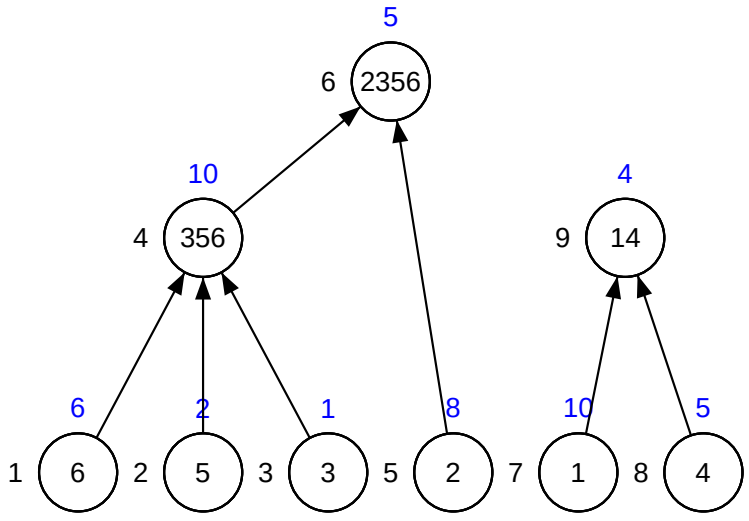
Merge $t = 7$.

$P(9) = \{((2,4),4),((2,1),6),((5,4),10)\}$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | (5,2) | (1,2) | (4,1) | (1,4) | nil |
| y | 6 | 2 | 1 | 10 | 8 | 5 | 10 | 5 | 0 |
| Next | nil | 1 | nil | nil | 4 | nil | 6 | 7 | 6 |
| Parent | 4 | 4 | 4 | 6 | 6 | nil | 9 | 9 | nil |
| Children | nil | nil | nil | (1,2,3) | nil | (4,5) | nil | nil | (7,8) |

# Primal iteration 9




| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | 9 | 6 | 6 | 9 | 6 | 6 |
| Leaf | 7 | 5 | 3 | 8 | 2 | 1 |

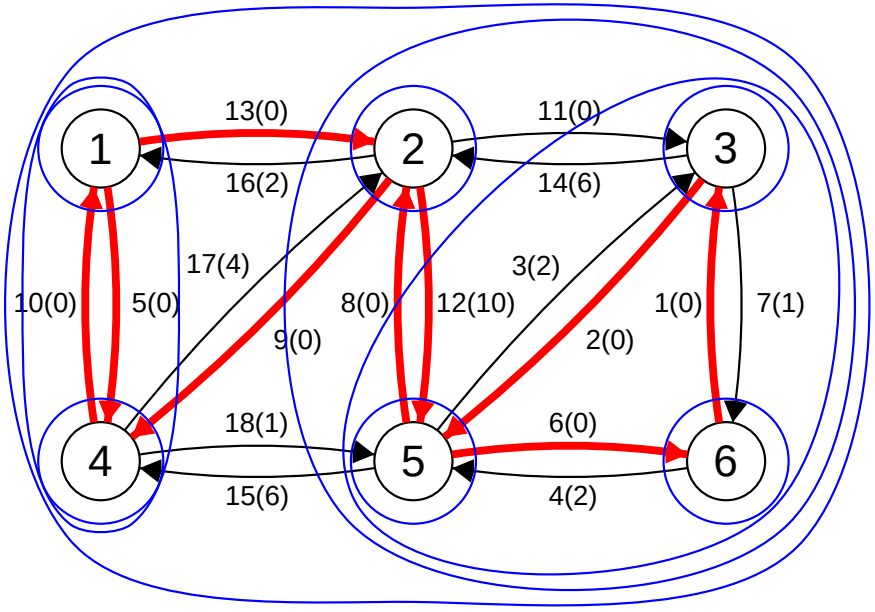
$k = 9 \quad s = 9$

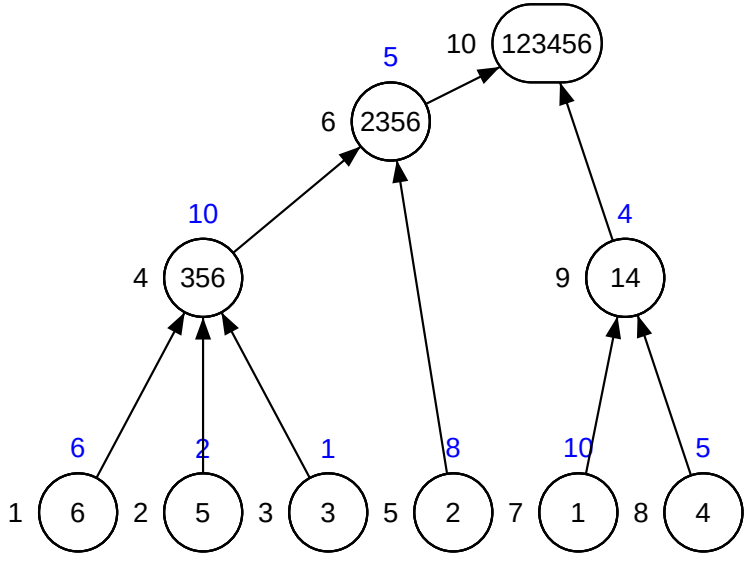
$(u, v) = (2, 4) \quad \overline{c}_{24} = 4$

$P(9) = \{((2, 1), 2), ((5, 4), 6)\}$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | (5,2) | (1,2) | (4,1) | (1,4) | (2,4) |
| y | 6 | 2 | 1 | 10 | 8 | 5 | 10 | 5 | 4 |
| Next | nil | 1 | nil | nil | 4 | nil | 6 | 7 | 6 |
| Parent | 4 | 4 | 4 | 6 | 6 | nil | 9 | 9 | nil |
| Children | nil | nil | nil | (1,2,3) | nil | (4,5) | nil | nil | (7,8) |

# Dual iteration 10




| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Comp | 10 | 10 | 10 | 10 | 10 | 10 |
| Leaf | 7 | 5 | 3 | 8 | 2 | 1 |

$k = 10 \quad s = 9$
Merge $t = 6$.
$P(10) =$
$\{((4,5),2),((2,1),2),((4,2),4),((5,4),6)\}$.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Arc | (5,6) | (3,5) | (6,3) | (2,5) | (5,2) | (1,2) | (4,1) | (1,4) | (2,4) | nil |
| y | 6 | 2 | 1 | 10 | 8 | 5 | 10 | 5 | 4 | 0 |
| Next | nil | 1 | nil | nil | 4 | nil | 6 | 7 | 6 | nil |
| Parent | 4 | 4 | 4 | 6 | 6 | 10 | 9 | 9 | 10 | nil |
| Children | nil | nil | nil | (1,2,3) | nil | (4,5) | nil | nil | (7,8) | (6,9) |

# Step 2

While Step 1 is independent of $r$, Step 2 produces different solutions depending on $r$.

---

$T \leftarrow \emptyset$
$RootList \leftarrow \emptyset$
$Dismantle(r)$
**while** $RootList \neq \emptyset$ **do**
  $s \leftarrow Extract(RootList)$
  $(u, v) \leftarrow Arc[s]$
  $T \leftarrow T \cup \{(u, v)\}$
  $Dismantle(v)$
**return** $T$

---

# Dismantle(v)

$s \leftarrow Leaf[v]$
**while** $Parent[s] \neq nil$ **do**
  **for** $t \in Children[Parent[s]] : t \neq s$ **do**
    $Parent[t] \leftarrow nil$
    $Insert(t, RootList)$
  $s \leftarrow Parent[s]$