Computational complexity

Combinatorial optimization

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Definitions

Optimization problems with discrete variables do not have a unique formulation.

Different formulations can be compared with one another.

A possible comparison criterion is the integrality gap, i.e. a measure of the distance between the optimal value of the problem and the optimal value of its continuous relaxation.

Here we consider integer linear problems, so that the feasible region of their continuous relaxation is a polyhedron.

Different formulations may yield different polyhedra when the integrality constraints are relaxed.

Ideal formulations

The ideal formulation has null integrality gap, i.e. its continuous relaxation is a linear programming problem with integer optimal solution.

The polyhedron describing the feasible region of the continuous relaxation of the ideal formulation is the convex hull of the integer feasible solutions.

It is very useful to characterize ideal formulations of integer linear programming problems, because when we know an ideal formulation we can solve the problem by the simplex algorithm (or any other algorithm for solving LPs) instead of having recourse to sophisticated and time-consuming techniques such as cutting planes, branch-and-bound etc.

Integrality of base solutions

Consider a linear program:

$$\begin{aligned} z_{LP} = \min \;\; & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}$$

where A and b are integer.

Consider now any base B. Basic variables have values

$$x^* = B^{-1} b$$

These values can be fractional only if the matrix B^{-1} has some fractional entry.

The inverse matrix

The inverse of the base matrix B is

$$B^{-1} = \frac{1}{det(B)} \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mm} \end{bmatrix}^T$$

where $a_{ij} = (-1)^{i+j} det(M_{ij})$ and M_{ij} is the sub-matrix of B obtained by deleting row i and column j.

Since B is integer, a_{ij} can be fractional only if $det(B)$ is different from ± 1.

Unimodularity

Definition. An integer matrix $A(n \times m)$ with $m \leq n$ is unimodular if any square submatrix $B(m \times m)$ is such that $det(B) \in \{-1, 0, 1\}$.

The feasible region of a linear program

$$\begin{aligned} z_{LP} = \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0, \end{aligned}$$

where A is unimodular and b is integer, is a polyhedron with integer vertices.

Standard form

Consider a LP set in standard form

$$
\begin{aligned}
z_{LP} = \min\ & c^T x \\
\text{s.t. } & Ax - Is = b \\
& x, s \geq 0.
\end{aligned}
$$

The constraint coefficients matrix is $[A, \ -I]$.

A canonical form is obtained by selecting m basic variables (columns). They can be either x or s variables. Hence the base sub-matrix B has the structure

$$
B = \begin{bmatrix} A' & -I' \\ A'' & 0 \end{bmatrix}^T
$$

where I' is an identity sub-matrix. Then:

$$
det(B) = \pm det(A'').
$$

Total unimodularity

Definition. An integer matrix A is totally unimodular (TUM) if any square sub-matrix B (of any size) is such that $det(B) \in \{-1, 0, 1\}$.

The feasible region of a linear program

$$\begin{aligned} z_{LP} = \min \ & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq 0, \end{aligned}$$

where A is totally unimodular and b is integer, is a polyhedron with integer vertices.

Characterization

A necessary condition for a matrix A to be TUM is to contain only entries in $\{-1, 0, 1\}$.

A sufficient condition for a matrix A to be TUM is the following:

1. every column has at most 2 non-zero entries;
2. there exists a partition (R_1, R_2) of the rows of A such that each column with two non-zero elements has these elements in different partitions if and only if they have the same sign.

An example

R_1	+				+		+	
		-						-
					-			
R_2			+			-	+	
				-				-
						+		

A sample matrix satisfying the sufficient condition.

Proof

The proof is by induction on the size k of the matrix.

If $k = 1$, then $A = [a_{ij}]$. Since $a_{ij} \in \{-1, 0, 1\}$, A is trivially TUM.

Consider a matrix A of size $k = k' + 1$.
Assume that A' is TUM for any sub-matrix A' of given size $k' \geq 1$.
Only three cases can occur:

- Case I: A has at least one column of null entries;
- Case II: A has no columns with null entries, but it has at least one column with a single non-zero entry;
- Case III: every column of A has two non-zero entries.

Proof

Case I. In this case $det(A) = 0$.

Case II. In this case A can be put in this form (by suitable permutations of rows and columns):

$$A = \begin{bmatrix} \pm 1 & * \ \dots \ * \\ 0 & \\ \dots & A' \\ 0 & \end{bmatrix}$$

$$det(A) = \pm det(A')$$

Since $det(A') \in \{-1, 0, 1\}$, then $det(A) \in \{-1, 0, 1\}$.

Proof

Case III. In this case it is easy to prove that $\det(A) = 0$, because the rows are not linearly independent.

R_1	+		+	
				-
	-			
R_2		-	+	
				-
		+		

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} = 0 \ \ \forall j.$$