# Linear programming and duality theory

## Combinatorial optimization

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# Linear Programming (LP)

A *linear program* is defined by

- linear constraints,
- a linear objective function.

Its variables are continuous and they can be

- free (unrestricted in sign),
- non-negative (constrained to be non-negative).

# LP: general form

In its general form, it reads like this:

$$
\begin{aligned}
P)\ \text{minimize } z &= c^T x \\
\text{subject to } a'x &\geq b' \\
a''x &= b'' \\
x' &\geq 0 \\
x'' &\text{ free}
\end{aligned}
$$

We can always get rid of the free variables by replacing each of them with the difference of two non-negative variables.

We can always get rid of linear equalities by variable elimination.

# LP: inequalities form

In its inequalities form, a linear program reads like this:

$$\begin{aligned} P)\ \text{minimize } z =& c^T x \\ \text{subject to } ax \geq& b \\ x \geq& 0 \end{aligned}$$

Every solution of the problem is a point in an $n$-dimensional space, where $n$ is the number of variables.

Solutions are *feasible* if the comply with all the constraints and the non-negativity restrictions. Otherwise they are *infeasible*.

The set of feasible solutions is the *feasible region* of the linear program.

# Constraints: geometrical interpretation

The constraints set of a linear program is made of linear inequalities.

In the variables space

- linear equalities correspond to *hyperplanes*;
- linear inequalities correspond to *halfspaces*.

The feasible region is a system of linear inequalities; it corresponds to the *intersection of halfspaces*, i.e. to a *polyhedron*.

Half spaces are convex; hence their intersection is also convex: polyhedra are convex.

**Theorem (Minkowsky and Weil).** Every point in a polyhedron can be obtained as a convex combination of its extreme points and its extreme rays.

# Polyhedra: three cases

Given a polyhedron one of these three cases occurs:

- the polyhedron is not empty and it is bounded (polytope);
- the polyhedron is not empty and it is not bounded;
- the polyhedron is empty.

# Case 1: polytope

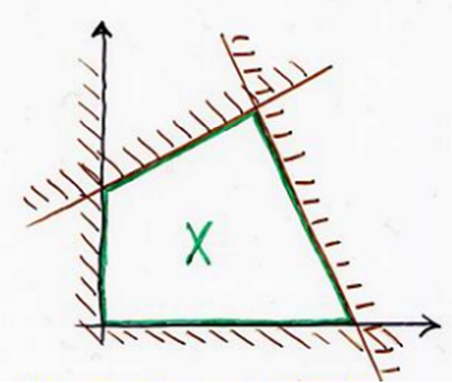


Figure: A polytope is defined by its extreme points.

# Case 2: unbounded polyhedron

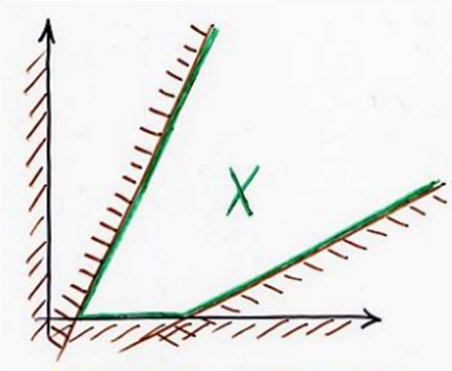


Figure: An unbounded polyhedron is defined by its extreme points and its extreme rays.

# Case 3: empty polyhedron

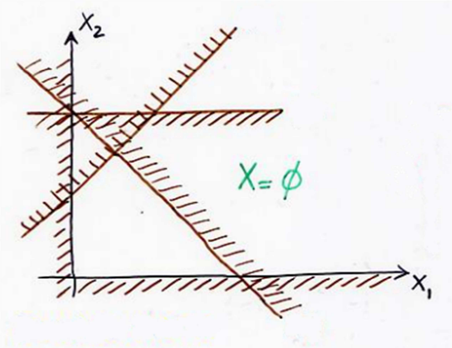


Figure: An empty polyhedron contains no points at all.

# Objective function: geometrical interpretation

The objective function of a linear program is represented by parallel hyperplanes, such that points on the same hyperplane have the same value.

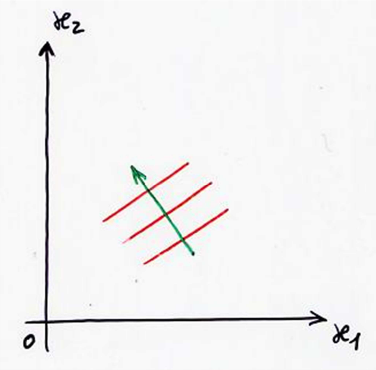


Figure: minimize $z = 2x_1 - 3x_2$

# Linear programs: three cases

Given a linear program $P$, one of these three cases occurs (and the simplex algorithm detects it in a finite number of steps):

- $P$ has a finite optimal solution;
- $P$ is unbounded;
- $P$ is infeasible.

# Case 1: finite optimal solution

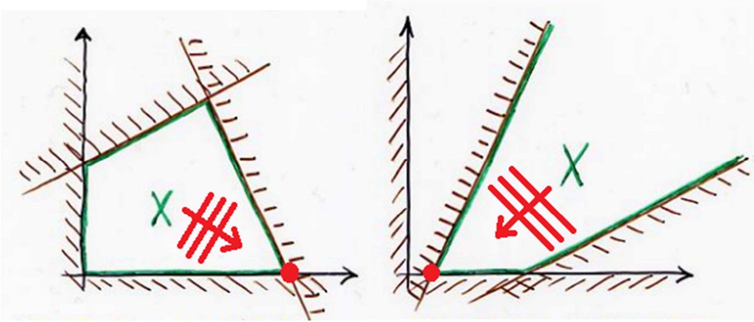


Figure: Two cases in which the linear program has a finite optimal solution.

# Case 2: unbounded problem

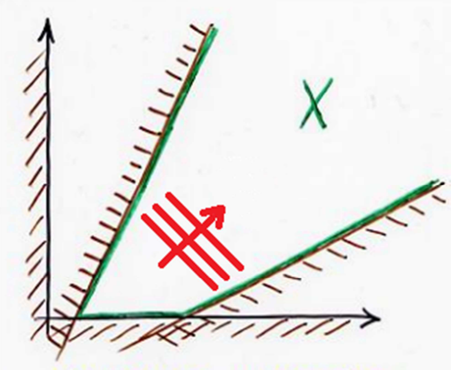


Figure: The linear program is unbounded if the direction of the objective function is in the cone defined by the extreme rays.

# Case 3: infeasible problem

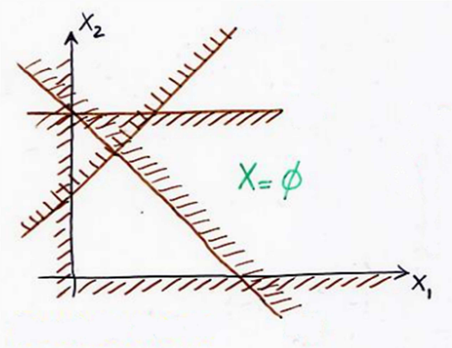


Figure: Empty polyhedron: the linear program in infeasible, independently of the objective function.

# LP: standard form

A linear program is in its standard form, when all inequalities have been turned into equalities by explicitly inserting non-negative *slack variables* or *surplus variables*.

Inequalities form:

$$
\begin{aligned}
P)\ \text{minimize } z = & \ c^T x \\
\text{subject to } & \ a'x \geq b' \\
& \ a''x \leq b'' \\
& \ x \geq 0.
\end{aligned}
$$

*m* inequality constraints
*n* non-negative variables.

Standard form:

$$
\begin{aligned}
P)\ \text{minimize } z = & \ c^T x \\
\text{subject to } & \ a'x - x^{surplus} = b' \\
& \ a''x + x^{slack} = b'' \\
& \ x, x^{surplus}, x^{slack} \geq 0.
\end{aligned}
$$

*m* equality constraints
$n + m$ non-negative variables.

The number of inequalities is the same in the two models.

# Bases

A *base* is a subset of $m$ linearly independent columns of the constraint matrix.

The $m$ variables corresponding to the columns of the base are *basic*.

The other $n$ variables are *non-basic*.

If the value of the non-basic variables is fixed, we are left with a non-degenerate system of $m$ linear equalities and $m$ variables, which provides a unique solution $x$.

If $x \geq 0$, then it is feasible. Otherwise it is infeasible.

# Base solutions

An inequality constraint is *active* iff its slack/surplus variable is null.

If we fix $n$ (non-basic) variables to 0, we define a solution $x$ in which $n$ constraints are active. The points of intersection of $n$ constraints in an $n$-dimensional variable space are called *base solutions*. They can be feasible or infeasible.

All the extreme points of the polyhedron are feasible base solutions.

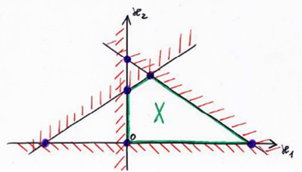


Figure: An example with $n = 2$, $m = 2$. There are 6 base solutions: 4 are feasible, 2 are infeasible.

# Linear Programming (LP) duality

Every linear program *P* has a dual linear program *D*.

$$
\begin{aligned}
P)\ \min z =\ & c'^T x' + c''^T x'' \\
\text{s.t. } & a'x' + a''x'' \geq b' && [y'] \\
& d'x' + d''x'' = b'' && [y''] \\
& x' \geq 0 \\
& x'' \text{ free}
\end{aligned}
$$

$$
\begin{aligned}
D)\ \max w =\ & b'^T y' + b''^T y'' \\
\text{s.t. } & a'^T y' + d'^T y'' \leq c' && [x'] \\
& a''^T y' + d''^T y'' = c'' && [x''] \\
& y' \geq 0 \\
& y'' \text{ free}
\end{aligned}
$$

# The fundamental theorem of LP duality

Given a primal-dual pair, one of these four cases occurs (and the simplex algorithm detects it in a finite number of steps):

- both $P$ and $D$ have a finite optimal solution;
- $P$ is unbounded and $D$ is infeasible;
- $D$ is unbounded and $P$ is infeasible;
- both $P$ and $D$ are infeasible.

# Weak duality theorem

**Weak duality theorem.**
For each feasible solution $x$ of $P$ and for each feasible solution $y$ of $D$, $z(x) \geq w(y)$.

**Corollary 1.**
If $P$ is unbounded, then $D$ is infeasible.

**Corollary 2.**
If $x$ is feasible for $P$ and $y$ is feasible for $D$ and $z(x) = w(y)$, then both $x$ and $y$ are also optimal.

# Strong duality theorem

**Strong duality theorem.**
If there exist a feasible and optimal solution $x^*$ for $P$ and a feasible and optimal solution $y^*$ for $D$, then $z(x^*) = w(y^*)$.

**Complementary slackness theorem.**
Given a feasible solution $x$ for $P$ and a feasible solution $y$ for $D$, necessary and sufficient condition for them to be optimal is:

- Primal complementary slackness conditions (they are $n$):

$$x' \left(c' - a'^T y' + d'^T y''\right) = 0$$

- Dual complementary slackness conditions (they are $m$):

$$y' \left(a' x' + a'' x'' - b'\right) = 0.$$

They only refer to inequality constraints, because those corresponding to equality constraints are always trivially satisfied.

# Primal-dual algorithms

A primal-dual algorithm solves linear programming problems exploiting duality theory and in particular the CSCs.

The algorithm is initialized with a dual feasible solution and a corresponding primal solution (in general, infeasible) satisfying the CSCs.

After every iteration the algorithm keeps a pair of primal (infeasible) and dual (feasible) solutions, satisfying the CSCs.

The algorithm alternates two types of iterations, and it monotonically decreases primal infeasibility until it achieves primal feasibility.

- Primal iteration: keeping the current dual feasible solution fixed, find a primal solution minimizing primal infeasibility among those satisfying the CSCs;
- Dual iteration: keeping the current primal solution fixed, modify the dual solution, keeping it feasible and the CSCs satisfied.