Minimum cost spanning r-arborescence

Doctoral course “Optimization on graphs” - Lecture 2.1

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Definitions

A digraph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$ is a spanning rooted out-arborescence (r-arborescence, for short) if and only if there is a unique directed path from its root node $r \in \mathcal{N}$ to all the other nodes in $\mathcal{N} \setminus \{r\}$ and no directed path from any node in $\mathcal{N} \setminus \{r\}$ to r.

The problem

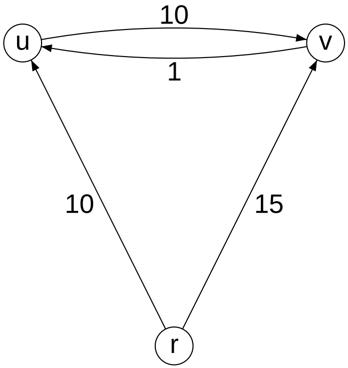
Problem data:

- a digraph $\mathcal{D} = (\mathcal{N}, \mathcal{A})$,
- a node $r \in \mathcal{N}$,
- a cost function $c : \mathcal{A} \to \Re_+$.

Problem (Minimum Cost Spanning *r*-Arborescence Problem).
Find a spanning *r*-arborescence of minimum cost.

Counter-example

The algorithms for the MSTP do not work.



A mathematical programming model

$$\min z = \sum_{a \in \mathcal{A}} c_a x_a$$

$$\text{s.t.} \sum_{a \in \delta^{in}(S)} x_a \geq 1 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

$$x_a \in \{0, 1\} \qquad \forall a \in \mathcal{A}$$

Integrality conditions are redundant.
We call r-cuts all arc subsets $\delta^{in}(S)$ corresponding to all node subsets S not containing r.

The dual model

$$\min z = \sum_{a \in \mathcal{A}} c_a x_a$$

$$\text{s.t.} \sum_{a \in \delta^{in}(S)} x_a \geq 1 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

$$x_a \geq 0 \qquad \forall a \in \mathcal{A}$$

This linear program has a dual.

$$\max w = \sum_{S \subseteq \mathcal{N} \setminus \{r\}} y_S$$

$$\text{s.t.} \sum_{S \subseteq \mathcal{N} \setminus \{r\} : a \in \delta^{in}(S)} y_S \leq c_a \qquad \forall a \in \mathcal{A}$$

$$y_S \geq 0 \qquad \forall S \subseteq \mathcal{N} \setminus \{r\}$$

Complementary slackness conditions

Primal C.S.C.: $x_a(c_a - \sum_{S \subseteq \mathcal{N} \setminus \{r\}: a \in \delta^{in}(S)} y_S) = 0 \quad \forall a \in \mathcal{A}$

Dual C.S.C.: $y_S(\sum_{a \in \delta^{in}(S)} x_a - 1) = 0 \quad \forall S \subseteq \mathcal{N} \setminus \{r\}$

The initial primal solution $x_a = 0 \; \forall a \in \mathcal{A}$ is primal infeasible (and super-optimal).
The corresponding dual solution $y_S = 0 \; \forall S \subseteq \mathcal{N} \setminus \{r\}$ is dual feasible (and sub-optimal).

Edmonds algorithm

This algorithm is due to Chu and Liu (1965), Edmonds (1967), Bock (1971).

Let us define $\mathcal{A}_0 = \{a \in \mathcal{A} : c_a = 0\}$. If $\mathcal{A}_0$ contains a spanning r-arborescence B, then B is a minimum cost spanning r-arborescence. Otherwise there is a s.c.c. K in the digraph $(\mathcal{N}, \mathcal{A}_0)$ such that

- $r \notin K$
- $c_a > 0 \ \forall a \in \delta^{in}(K)$.

Let us define $\alpha = \min\{c_a : a \in \delta^{in}(K)\}$. Modify the costs in this way: $c'_a := c_a - \alpha \ \forall a \in \delta^{in}(K)$ and $c'_a := c_a$ otherwise. Then search for a minimum cost spanning r-arborescence B with respect to the new cost function c'.

Edmonds algorithm: correctness

It is always possible to choose B such that it contains only one arc entering K, since K is strongly connected.

If $|B \cap \delta^{in}(K)| \geq 2$, then there exists a redundant arc $a \in B \cap \delta^{in}(K)$ such that $B \setminus \{a\} \cup \mathcal{A}_0$ still contains an r-arborescence B', with $c'(B') \leq c'(B) - c'_a \leq c'(B)$.

The optimal r-arborescence B chosen in this way is also optimal with respect to the original cost function c. For each other r-arborescence B', we have $c(B') \geq c(B)$:

- $c(B') = c'(B') + \alpha|B' \cap \delta^{in}(K)|$ by definition of the cost update procedure;
- $|B' \cap \delta^{in}(K)| \geq 1$ because of B' is a spanning r-arborescence;
- $c'(B') \geq c'(B)$, because B is optimal with respect to the modified costs c';
- $c'(B) + \alpha = c(B)$, because B contains only one arc entering K.

Edmonds algorithm: complexity (1)

The algorithm has complexity $O(nm)$, because it requires at most $2n$ iterations and each of them has complexity $O(m)$.

Let k be the number of s.c.c. in the digraph $(\mathcal{N}, \mathcal{A}_0)$. Let k_0 be the number of s.c.c. of $(\mathcal{N}, \mathcal{A}_0)$ with no zero-cost entering arcs.

At each iteration $k + k_0$ decreases by at least 1: if K remains a s.c.c., it now has at least one entering arc with zero cost, hence k_0 decreases; if K is merged with another s.c.c., then k decreases.

Initially $k = n - 1$ and $k_0 = n - 1$. Therefore the iterations are at most $2(n - 1)$.

Edmonds algorithm: complexity (2)

In $O(m)$ time it is possible to compute the set S of nodes that are not reachable from r in $(\mathcal{N}, \mathcal{A}_0)$.
In $O(m)$ time it is possible find the s.c.c. in the subgraph induced by S, sorting S in pre-topological order, so that its first node belong to a s.c.c. with no zero-cost entering arcs.

Therefore each iteration has time complexity $O(m)$.

Tarjan (1977): implementation in $O(\min\{n^2, m \log n\})$ time.

Dual ascent algorithms

A dual ascent algorithm iteratively does the following:

- Dual iteration: a violated primal constraint is selected (an r-cut S); the corresponding dual variable y_S enters the basis and is increased as much as possible, to activate a dual constraint corresponding to a primal variable x_a (an arc);
- Primal iteration: the corresponding primal variable x_a enters the basis and it is increased as little as possible in order to repair the infeasibility of the selected primal constraint (an r-cut).

Dual ascent

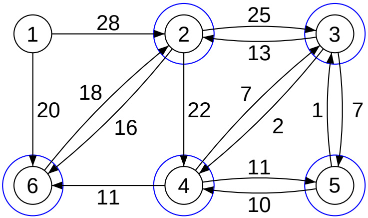
When a dual variable is increased (w improves), how much is "as much as possible"?

Dual feasibility requires $\sum_{S \subseteq \mathcal{N} \setminus \{r\}: a \in \delta^{in}(S)} y_S \leq c_a \quad \forall a \in \mathcal{A}$.
Therefore for each node subset S, the maximum value that the corresponding dual variable y_S can take is equal to the minimum reduced cost among all the arcs in the r-cut $\delta^{in}(S)$.

When a primal variable is increased (z worsens), how much is "as little as possible"?

The value 1 is the minimum amount to make at least one more primal constraint $\sum_{a \in \delta^{in}(S)} x_a \geq 1$ active. It is useless (sub-optimal) to give primal variables values larger than 1.

Edmonds algorithm: an example



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$ $\{2\}\{3\}\{4\}\{5\}\{6\}$ $k = 5$

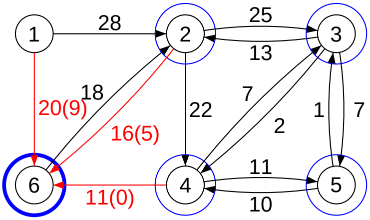
$S.C.C._0$ $\{2\}\{3\}\{4\}\{5\}\{6\}$ $k_0 = 5$

$k + k_0 = 10$.

$w = 0$.

$x = 0$.
$z = 0$.

Example: dual iteration 1



$x = 0.$
$z = 0.$

Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

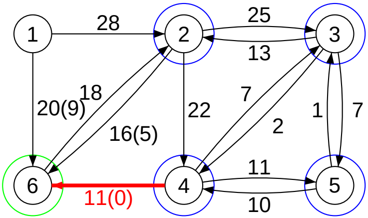
$S.C.C.$	$\{2\}\{3\}\{4\}\{5\}\{6\}$	$k = 5$
$S.C.C._0$	$\{2\}\{3\}\{4\}\{5\}\{6\}$	$k_0 = 5$

$k + k_0 = 10.$

Scan	$\{2\}\{3\}\{4\}\{5\}\{6\}$
End	$\{2\}\{3\}\{4\}\{5\}\{\mathbf{6}\}$

$S = \{6\}$
$\delta(S) = \{(1, 6), (2, 6), (4, 6)\}$
$\alpha = 11.$
$w = y_6 = 11.$

Example: primal iteration 1



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$ $\{2\}\{3\}\{4\}\{5\}\{6\}$ $k = 5$

$S.C.C._0$ $\{2\}\{3\}\{4\}\{5\}$ $k_0 = 4$

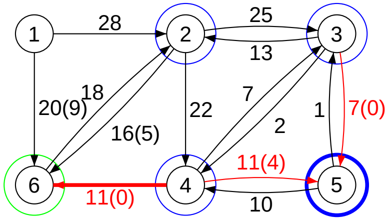
$k + k_0 = 9$.

$w = y_6 = 11$.

$x_{46} = 1$.

$z = 11$.

Example: dual iteration 2



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$ $\{2\}\{3\}\{4\}\{5\}\{6\}$ $k = 5$

$S.C.C._0$ $\{2\}\{3\}\{4\}\{5\}$ $k_0 = 4$

$k + k_0 = 9$.

Scan $\{2\}\{3\}\{4\}\{6\}\{5\}$

End $\{2\}\{3\}\{6\}\{4\}\{\mathbf{5}\}$

$x_{46} = 1$.

$z = c_{46} = 11$.

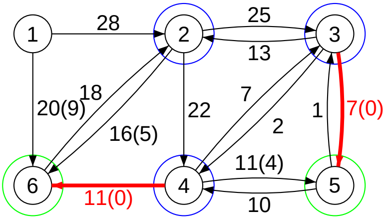
$S = \{5\}$

$\delta(S) = \{(3, 5), (4, 5)\}$

$\alpha = 7$.

$w = y_6 + y_5 = 18$.

Example: primal iteration 2



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$ $\{2\}\{3\}\{4\}\{5\}\{6\}$ $k = 5$

$S.C.C._0$ $\{2\}\{3\}\{4\}$ $k_0 = 3$

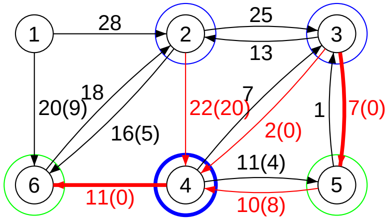
$k + k_0 = 8$.

$w = y_6 + y_5 = 18$.

$x_{46} = x_{35} = 1$.

$z = c_{46} + c_{35} = 18$.

Example: dual iteration 3



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$	$\{2\}\{3\}\{4\}\{5\}\{6\}$	$k = 5$
$S.C.C._0$	$\{2\}\{3\}\{4\}$	$k_0 = 3$

$k + k_0 = 8$.

Scan	$\{2\}\{3\}\{5\}\{4\}\{6\}$
End	$\{2\}\{5\}\{3\}\{6\}\{\mathbf{4}\}$

$x_{46} = x_{35} = 1$.

$z = c_{46} + c_{35} = 18$.

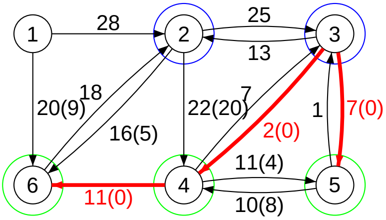
$S = \{4\}$

$\delta(S) = \{(2, 4), (3, 4), (5, 4)\}$

$\alpha = 2$.

$w = y_6 + y_5 + y_4 = 20$.

Example: primal iteration 3



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$ $\{2\}\{3\}\{4\}\{5\}\{6\}$ $k = 5$

$S.C.C._0$ $\{2\}\{3\}$ $k_0 = 2$

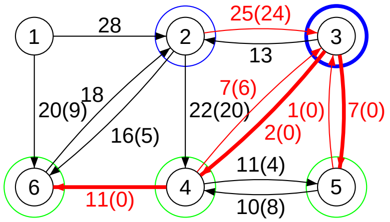
$k + k_0 = 7$.

$w = y_6 + y_5 + y_4 = 20$.

$x_{46} = x_{35} = x_{34} = 1$.

$z = c_{46} + c_{35} + c_{34} = 20$.

Example: dual iteration 4



$x_{46} = x_{35} = x_{34} = 1.$

$z = c_{46} + c_{35} + c_{34} = 20.$

Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

S.C.C.	$\{2\}\{3\}\{4\}\{5\}\{6\}$	$k = 5$
S.C.C.$_0$	$\{2\}\{3\}$	$k_0 = 2$

$k + k_0 = 7.$

Scan	$\{2\}\{3\}\{4\}\{6\}\{5\}$
End	$\{2\}\{6\}\{4\}\{5\}\{\mathbf{3}\}$

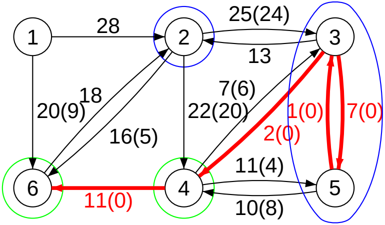
$S = \{3\}$

$\delta(S) = \{(2, 3), (4, 3), (5, 3)\}$

$\alpha = 1.$

$w = y_6 + y_5 + y_4 + y_3 = 21.$

Example: primal iteration 4



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$ $\{2\}\{3,5\}\{4\}\{6\}$ $k = 4$

$S.C.C._0$ $\{2\}\{3,5\}$ $k_0 = 2$

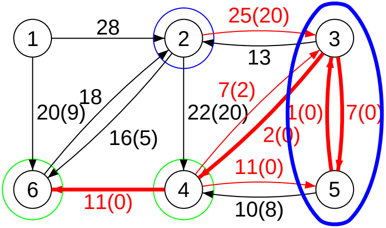
$k + k_0 = 6$.

$w = y_6 + y_5 + y_4 + y_3 = 21$.

$x_{46} = x_{35} = x_{34} = x_{53} = 1$.

$z = c_{46} + c_{35} + c_{34} + c_{53} = 21$.

Example: dual iteration 5



$x_{46} = x_{35} = x_{34} = x_{53} = 1.$

$z = c_{46} + c_{35} + c_{34} + c_{53} = 21.$

Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

S.C.C.	$\{2\}\{3,5\}\{4\}\{6\}$	$k = 4$
S.C.C.$_0$	$\{2\}\{3,5\}$	$k_0 = 2$

$k + k_0 = 6.$

Scan	$\{2\}\{3,5\}\{4\}\{6\}$
End	$\{2\}\{6\}\{4\}\{\mathbf{3}, \mathbf{5}\}$

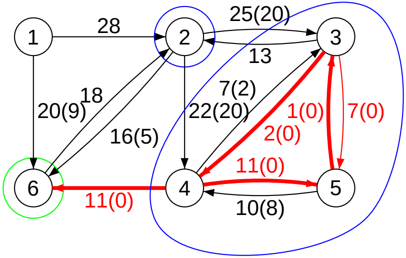
$S = \{3, 5\}$

$\delta(S) = \{(2,3), (4,3), (4,5)\}$

$\alpha = 4.$

$w = y_6 + y_5 + y_4 + y_3 + y_{35} = 25.$

Example: primal iteration 5



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$ $\{2\}\{3, 4, 5\}\{6\}$ $k = 3$

$S.C.C._0$ $\{2\}\{3, 4, 5\}$ $k_0 = 2$

$k + k_0 = 5$.

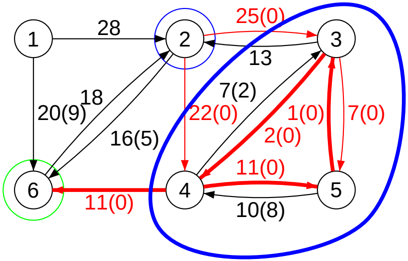
$w = y_6 + y_5 + y_4 + y_3 + y_{35} = 25$.

$x_{46} = x_{34} = x_{53} = x_{45} = 1$.

$z = c_{46} + c_{34} + c_{53} + c_{45} = 25$.

$x_{35} = 0$ is replaced by $x_{45} = 1$

Example: dual iteration 6



$x_{46} = x_{34} = x_{53} = x_{45} = 1.$

$z = c_{46} + c_{34} + c_{53} + c_{45} = 25.$

Either x_{23} or x_{24} enters the basis.

Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$	$\{2\}\{3,4,5\}\{6\}$	$k = 3$
$S.C.C._0$	$\{2\}\{3,4,5\}$	$k_0 = 2$

$k + k_0 = 5.$

Scan	$\{2\}\{3,4,5\}\{6\}$
End	$\{2\}\{6\}\{\mathbf{3},\mathbf{4},\mathbf{5}\}$

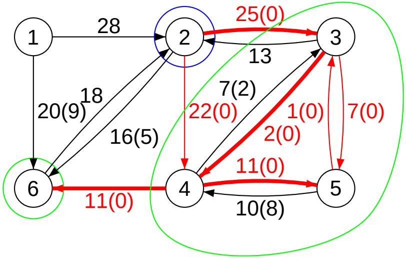
$S = \{3, 4, 5\}$

$\delta(S) = \{(2,3), (2,4)\}$

$\alpha = 20.$

$w = y_6 + y_5 + y_4 + y_3 + y_{35} + y_{345}$
$= 45.$

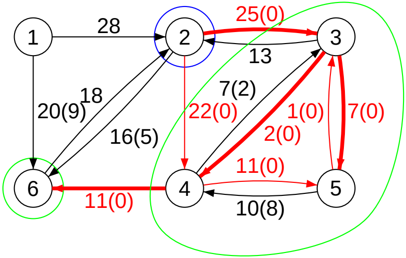
Option 1: x_{23} enters the basis



$x_{46} = x_{34} = x_{45} = x_{23} = 1.$

$z = c_{46} + c_{34} + c_{45} + c_{23} = 49.$

Wrong!



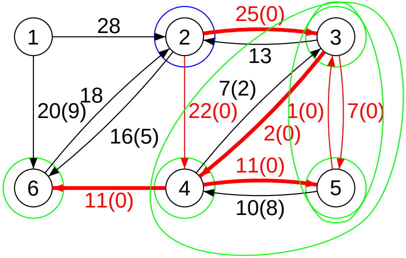
$x_{46} = x_{34} = x_{35} = x_{23} = 1.$

$z = c_{46} + c_{34} + c_{35} + c_{23} = 45.$

Right!

Why?

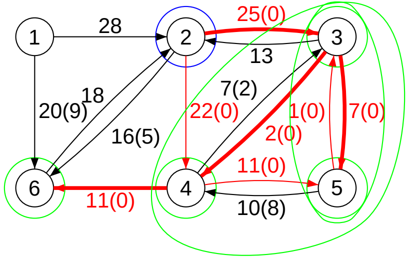
Option 1: x_{23} enters the basis



$x_{46} = x_{34} = \mathbf{x_{45}} = x_{23} = 1.$

$z = c_{46} + c_{34} + \mathbf{c_{45}} + c_{23} = 49.$

Wrong!



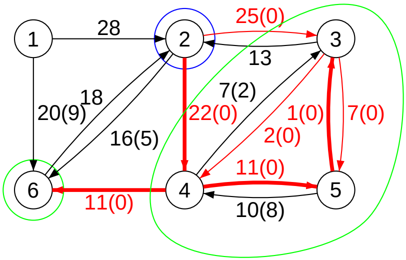
$x_{46} = x_{34} = \mathbf{x_{35}} = x_{23} = 1.$

$z = c_{46} + c_{34} + \mathbf{c_{35}} + c_{23} = 45.$

Right!

Because $y_{35} = 4$ and dual C.S.C. impose $y_{35}(x_{23} + x_{43} + x_{45} - 1) = 0$

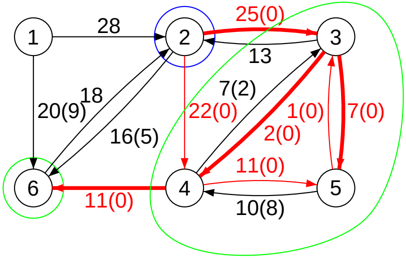
Option 2: x_{24} enters the basis



$x_{46} = x_{24} = x_{45} = x_{53} = 1$.
$z = c_{46} + c_{24} + c_{45} + c_{53} = 45$.

x_{34} is replaced by x_{24}.

Example: primal iteration 6



$x_{46} = x_{34} = x_{35} = x_{23} = 1.$
$z = c_{46} + c_{34} + c_{35} + c_{23} = 45.$

We choose option 1.

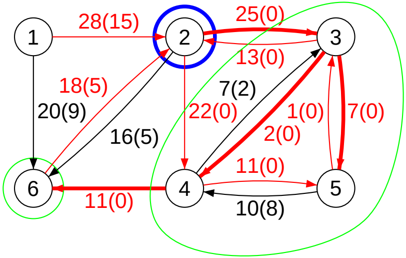
Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$	$\{2\}\{3, 4, 5\}\{6\}$	$k = 3$
$S.C.C._0$	$\{2\}$	$k_0 = 1$

$k + k_0 = 4.$

$w = y_6 + y_5 + y_4 + y_3 + y_{35} + y_{345}$
$= 45.$

Example: dual iteration 7



$x_{46} = x_{34} = x_{35} = x_{23} = 1$.

$z = c_{46} + c_{34} + c_{35} + c_{23} = 45$.

Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

S.C.C.	$\{2\}\{3,4,5\}\{6\}$	$k = 3$
S.C.C.$_0$	$\{2\}$	$k_0 = 1$

$k + k_0 = 4$.

Scan	$\{2\}\{3,4,5\}\{6\}$
End	$\{6\}\{3,4,5\}\{\mathbf{2}\}$

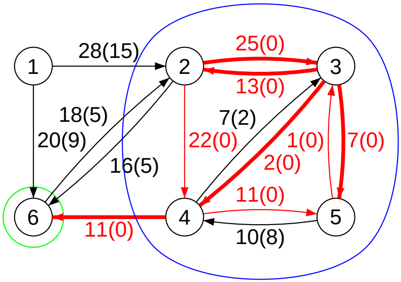
$S = \{2\}$

$\delta(S) = \{(1,2), (3,2), (6,2)\}$

$\alpha = 13$.

$w = y_6 + y_5 + y_4 + y_3 + y_{35} + y_{345} +$
$+ y_2 = 58$.

Example: primal iteration 7



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$ $\{2, 3, 4, 5\}\{6\}$ $k = 2$

$S.C.C._0$ $\{2, 3, 4, 5\}$ $k_0 = 1$

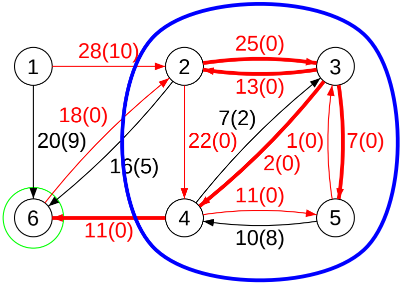
$k + k_0 = 3$.

$$w = y_6 + y_5 + y_4 + y_3 + y_{35} + y_{345} + + y_2 = 58.$$

$$x_{46} = x_{34} = x_{35} = x_{23} = x_{32} = 1.$$

$$z = c_{46} + c_{34} + c_{35} + c_{23} + c_{32} = 58.$$

Example: dual iteration 8



Unreachable from 1: $\{2,3,4,5,6\}$.

$S.C.C.$ $\{2,3,4,5\}\{6\}$ $k=2$

$S.C.C._0$ $\{2,3,4,5\}$ $k_0=1$

$k+k_0=3$.

$Scan$ $\{2,3,4,5,6\}\{6\}$

End $\{6\}\{\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\}$

$x_{46}=x_{34}=x_{35}=x_{23}=x_{32}=1$.

$z=c_{46}+c_{34}+c_{35}+c_{23}+c_{32}=58$.

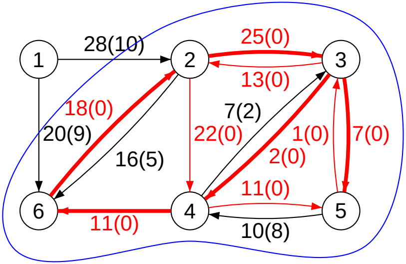
$S=\{2,3,4,5\}$

$\delta(S)=\{(1,2),(6,2)\}$

$\alpha=5$.

$$w=y_6+y_5+y_4+y_3+y_{35}+y_{345}+ \\ +y_2+y_{2345}=63.$$

Example: primal iteration 8



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$ $\{2, 3, 4, 5, 6\}$ $k = 1$

$S.C.C._0$ $\{2, 3, 4, 5, 6\}$ $k_0 = 1$

$k + k_0 = 2.$

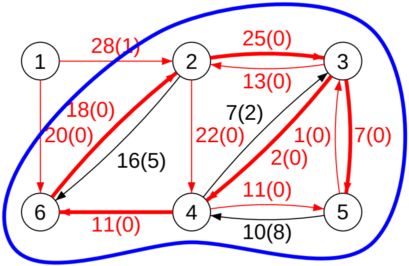
$$w = y_6 + y_5 + y_4 + y_3 + y_{35} + y_{345} + \\ + y_2 + y_{2345} = 63.$$

$x_{46} = x_{34} = x_{35} = x_{23} = x_{62} = 1.$

$z = c_{46} + c_{34} + c_{35} + c_{23} + c_{62} = 63.$

x_{62} replaces x_{32}.

Example: dual iteration 9



Unreachable from 1: $\{2, 3, 4, 5, 6\}$.

$S.C.C.$	$\{2, 3, 4, 5, 6\}$	$k = 1$
$S.C.C._0$	$\{2, 3, 4, 5, 6\}$	$k_0 = 1$

$k + k_0 = 2$.

Scan	$\{2, 3, 4, 5, 6\}$
End	$\{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$

$x_{46} = x_{34} = x_{35} = x_{23} = x_{62} = 1$.

$z = c_{46} + c_{34} + c_{35} + c_{23} + c_{62} = 63$.

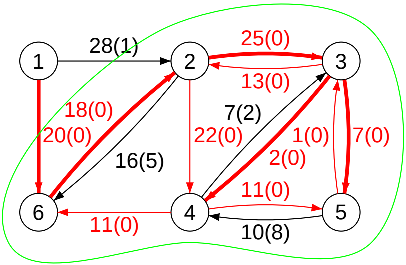
$S = \{2, 3, 4, 5, 6\}$

$\delta(S) = \{(1, 2), (1, 6)\}$

$\alpha = 9$.

$$w = y_6 + y_5 + y_4 + y_3 + y_{35} + y_{345} + y_2 + y_{2345} + y_{23456} = 72.$$

Example: primal iteration 9



Unreachable from 1: $\{\}$.

$S.C.C.$ $k = 0$

$S.C.C._0$ $k_0 = 0$

$k + k_0 = 0$.

Algorithm termination.

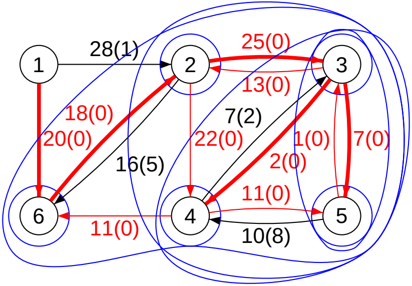
$$w = y_6 + y_5 + y_4 + y_3 + y_{35} + y_{345} + \\ + y_2 + y_{2345} + y_{23456} = 72.$$

$x_{34} = x_{35} = x_{23} = x_{62} = x_{16} = 1$.

$z = c_{34} + c_{35} + c_{23} + c_{62} + c_{16} = 72$.

x_{16} replaces x_{46}.

Example: optimal solution



$y_6 = 11$
$y_5 = 7$
$y_4 = 2$
$y_3 = 1$
$y_{35} = 4$
$y_{345} = 20$
$y_2 = 13$
$y_{2345} = 5$
$y_{23456} = 9$

$x_{34} = x_{35} = x_{23} = x_{62} = x_{16} = 1.$
$z = c_{34} + c_{35} + c_{23} + c_{62} + c_{16} = 72.$

$w = y_6 + y_5 + y_4 + y_3 + y_{35} + y_{345} +$
$+ y_2 + y_{2345} + y_{23456} = 72.$

The reduced cost of each arc is the difference between the original cost and the sum of all dual variables corresponding to the s.c.c. reached by the arc.