Optimization problems in logistics: routing

Logistics

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Routing

Routing problems are classical optimization problems arising in short-haul transportation.

They typically correspond to tactical or operational level decisions.

One can assume that facilities such as vehicle depots have already been selected and the appropriate mix of types of vehicles have been decided; the remaining problem is to decide the optimal route that each vehicle must follow to perform pickup/delivery operations.

The resulting mathematical models are usually integer linear programming, that can be very difficult to solve and therefore require specific and sophisticated optimization algorithms.

Several heuristic methods have also been devised for solving routing problems approximately.

Geographical data

In short-haul transportation at least origins or destinations, if not both, are located at specific customer sites, not necessarily at logistic hubs or inter-modal stations (often transformed into vertices of as suitable shortest paths graph).

Therefore, routing problems usually require the knowledge of a weighted (di-)graph representing the road/street network of the region or city of interest in full detail (lat/lon coordinates of each address, one-way streets, forbidden manoeuvres, real-time traffic information, compatibility between streets and type of vehicle,...).

These data are now available as digital information, but they are (very) expensive.
Open data: Open Street Map.

Some of them are available for free as answers to queries to a centralized server, but with a limit on the number of queries per unit of time.

Geographical Information Systems have not been designed to support optimization algorithms: data format conversion is usually

Graphs

Digital data from GISs are often used to define suitable auxiliary graphs in which

- vertices/nodes correspond to the locations to be visited;
- edges/arcs correspond to shortest paths on the original street graph;
- a traversal time, a length and a cost are associated with each edge/arc;
- consequently, the graph is complete and the triangle inequality holds.

Classification

A possible taxonomy of routing problems is the following:

- node routing, where locations must be visited (points in a graph): pick-up and delivery of goods;
- arc routing, arcs must be traversed: street cleaning, mail delivery, garbage collection, snow removal....
- single-vehicle or multi-vehicle;
- static or dynamic (real-time data);
- freight or passengers transportation;
- deterministic or stochastic;
- symmetric or asymmetric graphs;
- split delivery or not;
- with or without time window constraints, capacity constraints, maximum route duration/length constraints, precedence constraints,...
- integrated with inventory, location, scheduling,...

The typical objective to be optimized is the total distance traveled.

The Asymmetric TSP

The Asymmetric Traveling Salesman Problem is the problem of finding a minimum cost Hamiltonian circuit on a given weighted digraph $D = (N, A)$.

Binary variables are associated with arcs: $x_{ij} = 1$ if and only if arc (i, j) belongs to the solution.

$$\begin{aligned}
\text{minimize } z = & \sum_{(i,j)\in A} c_{ij} x_{ij} & \\
\text{s.t. } & \sum_{j\in N} x_{ij} = 1 & \forall i \in N \\
& \sum_{i\in N} x_{ij} = 1 & \forall j \in N \\
& S.E.C. & \\
& x_{ij} \in \{0, 1\} & \forall (i, j) \in A
\end{aligned}$$

Sub-tour Elimination Constraints are needed to forbid solutions made of disjoint sub-tours.

Sub-tour elimination

Sub-tour Elimination Constraints can be formulated in different ways.

- No proper subset of N can contain a cycle:

$$\sum_{(i,j)\in A(S)} x_{ij} \leq |S| - 1 \ \ \forall S \subset N : 2 \leq |S| \leq N/2.$$

- All oriented cuts must be traversed at least once:

$$\sum_{(i,j)\in (S,\overline{S})} x_{ij} \geq 1 \ \ \forall S \subset N : |S| \geq 2.$$

- Each node must appear in a different position along the circuit:

$$u_i - u_j + nx_{ij} \leq n - 1 \ \ \forall 2 \leq i \neq j \leq n$$

  with (integer) variables $0 \leq u_i \leq n - 1$ for each node.

When S.E.C. are exponential in number, they are generated at run-time (cutting planes algorithms).

The Symmetric TSP

The Symmetric Traveling Salesman Problem is the problem of finding a minimum cost Hamiltonian cycle in a given weighted graph $G = (V, E)$.

Binary variables are associated with edges: $x_{ij} = 1$ if and only if edge $[i, j]$ belongs to the solution.

$$
\begin{aligned}
\text{minimize } z = \sum_{[i,j] \in E} c_{ij} x_{ij} & \\
\text{s.t. } \sum_{j \in \delta(i)} x_{ij} = 2 & \qquad \forall i \in V \\
S.E.C. & \\
x_{ij} \in \{0, 1\} & \qquad \forall (i, j) \in E
\end{aligned}
$$

Sub-tour Elimination Constraints are needed again to forbid solutions made of disjoint cycles.

The (A)VRP

The (asymmetric) Vehicle Routing Problem is the problem of finding a set of routes to be assigned to the vehicles of a given fleet so that:

- all nodes are visited by a route;
- all routes start and end at a special node (depot);
- the number of routes is not larger than the given number of vehicles;
- possibly some additional constraints are satisfied.

The formulation requires variables representing assignment and routing decisions.

The Chinese postman problem

Given a weighted graph, find a minimum cost Euler tour on it.

The problem can be efficiently solved as follows (symmetric graph version):

1. find all vertices with odd degree;
2. compute a minimum cost matching between them (on the shortest paths graph);
3. find an Euler tour on the resulting multi-graph.

On asymmetric digraphs the problem of re-balancing the in-degree and out-degree of all nodes is efficiently solved as a min cost flow problem.