Inventory management

Logistics

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Inventory systems

In the supply chain there are several points where stocks are kept for different purposes.

These are all goods and products that do not undergo transformation, assembly or similar operations. Stocks are placed

- in the production system (e.g. between a machine and another),
- in the distribution system (e.g. travelling stocks, stocks in the shops),
- between a facility and another within the supply chain (e.g. warehouses, wholesalers).

Costs

Costs due to stocks are mainly of three types:

- costs for purchasing;
- costs for holding the inventory;
- costs for obsolescence.

Stocked goods can be classified on the basis of:

- their value for unit of weight or volume;
- the frequency with which they are requested;
- the predictability/uncertainty of their demand.

Classification

Inventory systems can be classified with four main criteria:

- number of inventory locations,
- number of product types,
- deterministic or non-deterministic,
- replenishment modality (continuous or discrete).

1/1/D/C systems: data

Single-point, single-product, deterministic inventory systems with continuous replenishment are characterized by:

- a demand d;
- a replenishment rate r ($r > d$);
- a replenishment period T_r;
- a period T;
- a maximum level M;
- a maximum stock-out level s.

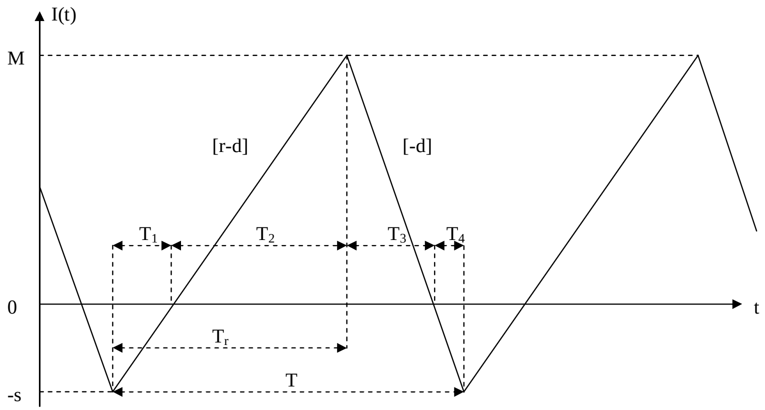
During the replenishment periods the level of the inventory increases at a rate $r - d$; during the other periods if decreases at a rate d.

The period T and the order quantity q that is replenished in each replenishment period are linked with d and r through the relation

$$q = dT = rT_r.$$

The value of q (or T) is a decision variable.

1/1/D/C systems: graphical representation



The extreme values *M* and *s* can also be decision variables.

1/1/D/C systems: costs

We indicate the overall cost function (per unit of time) as follows:

$$\mu(q, s) = \frac{1}{T}(k + cq + h\overline{I}T + us + v\overline{S}T)$$

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- k is the fixed cost for each replenishment operation;
- c is the price of the product;
- h is the obsolescence cost per unit of product and per unit of time;
- u is the unitary stock-out cost;
- v is the unitary stock-out cost per unit of product and per unit of time;
- $\overline{I}$ is the average inventory level;
- $\overline{S}$ is the average stock-out level.

1/1/D/C systems: analysis

The following relations hold:

- $T_r = T_1 + T_2$
- $T = T_1 + T_2 + T_3 + T_4$
- $s + M = (r - d)T_r$
- $\bar{I} = \frac{1}{T}\frac{M(T_2+T_3)}{2}$
- $\overline{S} = \frac{1}{T}\frac{s(T_1+T_4)}{2}$
- $s = (r - d)T_1$
- $M = (r - d)T_2$
- $M = dT_3$
- $s = dT_4$.

From them we obtain

$$\mu(q, s) = \frac{kd}{q} + cd + \frac{h[q(1 - \frac{d}{r}) - s]^2}{2q(1 - \frac{d}{r})} + \frac{usd}{q} + \frac{vs^2}{2q(1 - \frac{d}{r})}.$$

1/1/D/C systems: solution

By computing the partial derivatives of $\mu(q, s)$ wth respect to the two variables q and s and imposing they are null, we obtain:

$$q^* = \sqrt{\frac{h+v}{v}\left(\frac{2kd}{h(1-d/r)} - \frac{(ud)^2}{h(h+v)}\right)}$$

and

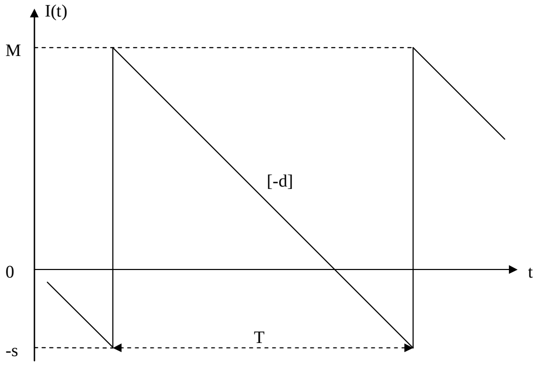
$$s^* = \frac{(hq^* - ud)(1-d/r)}{h+v}$$

In the case with no stock-out allowed ($s = 0$), they reduce to:

$$q^* = \sqrt{\frac{2kd}{h(1-d/r)}}.$$

1/1/D/D systems: graphical representation

We can study them as special cases of 1/1/D/C systems when $r \to \infty$.



1/1/D/D systems: solution

Then we have $\mu(q,s) = \frac{kd}{q} + cd + \frac{h(q-s)^2}{2q} + \frac{usd}{q} + \frac{vs^2}{2q}$ from which

$$q^* = \sqrt{\frac{h+v}{v}\left(\frac{2kd}{h} - \frac{(ud)^2}{h(h+v)}\right)}$$

$$s^* = \frac{(hq^* - ud)}{h+v}.$$

In the case with no stock-out allowed ($s = 0$) we have

$$q^* = \sqrt{\frac{2kd}{h}}$$

The corresponding value of the minimum cost is

$$\mu(q^*) = \sqrt{2kdh} + cd.$$

The optimal value q^* is also called Economic Order Quantity (EOQ).

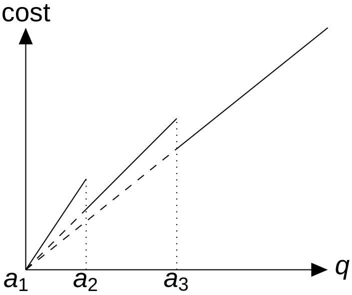
Models with discounts

The optimal order quantity may also depend from additional factors like the possibility of obtaining discounts on the purchase price in case of large orders.

We consider two different types of discounts:

- total quantity discount;
- incremental discount.

Total quantity discount



The cost for purchasing the product is $c_i q$ for $a_i \leq q \leq a_{i+1}$ with $i = 1, \dots, n$.

The unit cost (price) c_i decreases when i increases:

- $a_i > a_{i-1} \ \ \forall i = 2, \dots, n$
- $c_i < c_{i-1} \ \ \forall i = 2, \dots, n$

Remark: A larger quantity may cost less than a smaller one.

Solution

For each price range $i = 1, \ldots, n$:

- determine the EOQ $\overline{q}_i$ as usual;
- set

$$q_i^* = \begin{cases} a_i & \text{if } \overline{q}_i < a_i \\ \overline{q}_i & \text{if } a_i \leq \overline{q}_i \end{cases}$$

- compute the corresponding cost $\mu(q_i^*)$;
- choose the price range for which the cost is minimum.

Mathematical formulation (1/3)

The problem can be solved with no computations, by formulating its *mathematical model*.

Data:

- the minimum amount a_i for each range $i = 1, \ldots, n$;
- the price c_i for each range $i = 1, \ldots, n$.

Variables:

- the quantity $q \geq 0$ purchased;
- binary variables x_i indicating whether price range $i = 1, \ldots, n$ is selected.

Mathematical formulation (2/3)

Constraints:

- One of the price ranges must be selected:

$$\sum_{i=1}^{n} x_i = 1$$

- The order quantity must be large enough to have the selected discount:

$$q \geq a_i x_i \quad \forall i = 1, \ldots, n.$$

Mathematical formulation (3/3)

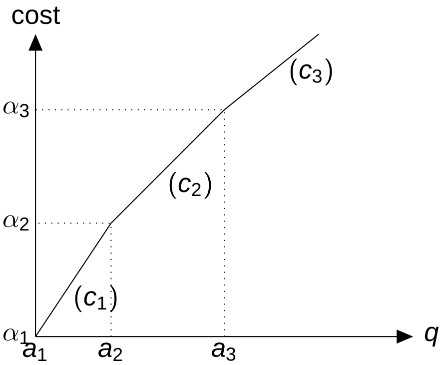
The price is $c = \sum_{i=1}^{n} c_i x_i$ (only one term is non-zero).

Objective: minimize the overall cost.

$$\text{minimize } \mu = \frac{kd}{q} + d\sum_{i=1}^{n} c_i x_i + \frac{pq}{2}\sum_{i=1}^{n} c_i x_i.$$

Once formulated in this way, in terms of data, variables, constraints and objective function, the problem can be solved by a non-linear binary programming solver.

Incremental discounts

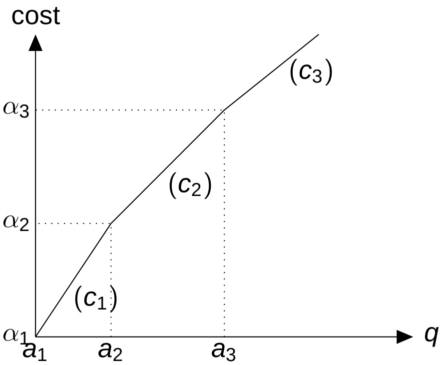


The price c_i decreases when i increases:

- $a_i > a_{i-1} \ \ \forall i = 2, \ldots, n-1,$
- $c_i < c_{i-1} \ \ \forall i = 2, \ldots, n.$

In this case the cost is monotonically increasing with q.

Incremental discounts



The cost for purchasing the product is $c(q)q = \alpha_i + c_i(q - a_i)$ for each range $a_i \leq q \leq a_{i+1}$ with $i = 1, \ldots, n$.

Analysis

The cost expression

$$\mu(q) = \frac{kd}{q} + cd + \frac{hq}{2}$$

must be modified (remembering that $h = pc$):

$$\mu(q) = \frac{kd}{q} + c(q)d + \frac{pc(q)q}{2}.$$

Putting in evidence the purchase cost $c(q)\,q$, we get:

$$\mu(q) = \frac{kd}{q} + c(q)\,q\frac{d}{q} + \frac{p}{2}c(q)\,q.$$

For each range $i = 1, \ldots, n$ we can replace

$$c(q)\,q = \alpha_i + c_i(q - a_i)$$

and we get

$$\mu_i(q) = (k + \alpha_i - c_i a_i)\frac{d}{q} + c_i d + \frac{p}{2}(\alpha_i - c_i a_i) + \frac{pc_i q}{2}.$$

Analysis

The expression

$$\mu_i(q) = (k + \alpha_i - c_i a_i)\frac{d}{q} + c_i d + \frac{p}{2}(\alpha_i - c_i a_i) + \frac{pc_i q}{2}.$$

is analogous to the one with constant price

$$\mu(q) = \frac{kd}{q} + cd + \frac{hq}{2}.$$

To obtain the other we must:

- replace k with $k + \alpha_i - c_i a_i$;
- replace cd with $c_i d + \frac{p}{2}(\alpha_i - c_i a_i)$;
- replace h with pc_i.

Solution

Hence in the computation of the EOQ, instead of $\sqrt{\frac{2kd}{h}}$ we have $\sqrt{\frac{2(k+\alpha_i-c_ia_i)d}{pc_i}}$.

For each range $i = 1, \ldots, n$:

- compute $\overline{q}_i = \sqrt{\frac{2(k+\alpha_i-c_ia_i)d}{pc_i}}$;
- discard $\overline{q}_i$ if it falls outside the range $[a_i, a_{i+1}]$;
- select the range for which the cost $\mu_i(\overline{q}_i)$ is minimum.

Mathematical formulation (1/3)

Also in this case the problem can be solved by formulating its *mathematical model*.

Data:

- the demand d;
- the fixed order cost k;
- the obsolescence rate p;
- the minimum required amount a_i for each range $i = 1, \ldots, n$;
- the price c_i for each range $i = 1, \ldots, n$.

We define the cost α_i for saturating all ranges before range i:

$$\alpha_i = \sum_{j=1}^{i-1} c_j (a_{j+1} - a_j).$$

Mathematical formulation (2/3)

Variables:

- the order quantity in the selected range, $r \geq 0$;
- binary variables x_i indicating the last used range;
- the order quantity $q \geq 0$;
- its cost C.

Objective: minimize the overall cost.

$$\text{minimize } \mu = \frac{kd}{q} + \frac{dC}{q} + \frac{pC}{2}.$$

Mathematical formulation (3/3)

Constraints:

- One range must be selected as the last one:

$$\sum_{i=1}^{n} x_i = 1.$$

- The order quantity depends on r and the selected last range:

$$q = \sum_{i=1}^{n} a_i x_i + r.$$

- The order cost depends on r and the selected last range:

$$C = \sum_{i=1}^{n} \alpha_i x_i + c_i r.$$

A binary non-linear programming model is obtained.

Multi-product inventory systems

Keeping multiple types of products in the same inventory system has pros and cons.

The main drawback consists in the need of complying with limited shared resources: budget, capacity, etc.

The main advantage comes from the possibility of combining multiple orders, producing economies of scale.

Capacity constraints

A trivial way of managing a multi-product system would be to consider each product type separately and to optimize order quantities and periods independently from one another.

Such a policy can be infeasible because of constraints on shared resources. For instance:

- the overall capital corresponding to the inventory must not be larger than a given limit;
- the overall volume of the inventory cannot be larger than a given capacity of the warehouse.

In general we indicate these constraints as *capacity constraints*.

Solution

A practial way of complying with capacity constraints is to artificially increase the value of the obsolescence percentage p, if it is the same for all product types, so as to decrease the order quantities q (and consequently the average inventory), until all capacity constraints are satisfied.

A better method is to solve the mathematical model of the cost minimization problem after the insertion of the capacity constraints.

Synchronization

To obtain scale economies in transportation, it is often convenient to order two or more products simultaneously.

When this occurs, it may be convenient to adopt an order frequency which is slightly different from the optimal one, if this allows for synchronization.

For instance in the case of two products, let indicate with

- T, the period in which we issue a joint order,
- N_1 and N_2, the number of orders of the two products in each period of length T,
- k_{12}, the cost of a joint order ($k_{12} < k_1 + k_2$).

Synchronization

The costs of a single-product inventory system as a function of T (instead of q) is:

$$\mu(T) = \frac{k}{T} + cd + \frac{hdT}{2}.$$

In the multi-product case, it becomes:

$$\mu(T, N_1, N_2) = \frac{k_{12} + (N_1 - 1)k_1 + (N_2 - 1)k_2}{T} + c_1 d_1 + c_2 d_2 + \frac{h_1 d_1 T}{2N_1} + \frac{h_2 d_2 T}{2N_2}$$

and it is minimized when

$$T^*(N_1, N_2) = \sqrt{\frac{2N_1 N_2 (k_{12} + (N_1 - 1)k_1 + (N_2 - 1)k_2)}{h_1 d_1 N_2 + h_2 d_2 N_1}}.$$

As an alternative to the computation of T^* by iterative trials with different values of N_1 and N_2, one can use a solver to find the optimal solution of the mathematical model, after expressing the cost μ as a function of the integer variables N_1 and N_2 and the variable T.

Non-deterministic models

To make models more realistic, it is convenient to study systems in which some data are not known with certainty, but only as probability distributions.

Two typical cases are the demand d and the lead time τ.
The former can be affected by several factors (economic cycles, actions of the competitors and the customers,...).
The latter can be affected by delays in transportation.

Assumptions:

- d has normal distribution with avg. value $\overline{d}$ and variance σ_d^2, stationary;
- τ has normal distribution with avg. value $\overline{\tau}$ and variance σ_{tau}^2, stationary;
- d and τ are independent.

Policies

Non-determinism in d or τ affects the time when the orders are done.

In deterministic models with no stock-out allowed, we have $l = d\tau$, where l is the reorder point, i.e. the level of the inventory that triggers a new order.
In non-deterministic inventory systems, this could lead to stock-out, because of a peak in demand or a delay in the lead time.

Therefore it is necessary to monitor the level of the inventory and to keep an additional *safety stock*, to comply with uncertainty.
This can be done in several ways. The most common policies are:

- fixed reorder level;
- fixed reorder period;
- (s, S) method.

Fixed reorder level

This method implies continuous monitoring of the inventory level. When it reaches level l, then a new order is issued.

The EOQ is chosen according to the average demand:

$$q^* = \sqrt{\frac{2k\bar{d}}{h}}.$$

The reorder point l must be chosen so that the probability of stock-out is within a certain threshold parameter.

Fixed reorder level

Given a random variable X with standard normal probability distribution, we indicate with z_α the value such that

$$P(X < z_\alpha) = \alpha.$$

Hence we set

$$I = \overline{d}\tau + z_\alpha \sigma_d \sqrt{\tau} \quad \text{if } \tau \text{ is constant.}$$

$$I = \overline{d}\overline{\tau} + z_\alpha \sqrt{\sigma_d^2 \overline{\tau} + \sigma_\tau^2 \overline{d}^2} \quad \text{if } \tau \text{ is random.}$$

The red term is the safety stock.

Fixed reorder period

This method only requires to sample the inventory level periodically. Every time an order is issued. The ordered quantity is not always the same, but it is computed as the difference between a target level and the current level.

$$q(t_i) = S - I(t_i).$$

The period is computed according the usual EOQ formula as in the deterministic case:

$$T^* = \sqrt{\frac{2k}{h\overline{d}}}.$$

Fixed reorder period

The target level S is chosen so that the probability of stock-out is within a certain threshold parameter.

$$S = \overline{d}(T^* + \tau) + z_\alpha \sigma_d \sqrt{T^* + \tau} \quad \text{if } \tau \text{ is constant.}$$

$$S = \overline{d}(T^* + \overline{\tau}) + z_\alpha \sqrt{\sigma_d^2 (T^* + \overline{\tau}) + \sigma_\tau^2 \overline{d}^2} \quad \text{if } \tau \text{ is random.}$$

The red term is the safety stock: it depends on the variance of d and τ.

Compared to the previous policy, the safety stock is larger but the sampling cost is lower.

(s, S) policy

This policy is a trade-off between the previous two.

We set two levels:

$$s = (T^* + \overline{\tau})\overline{d}$$

$$S = s + q^* - \frac{\overline{d}T^*}{2}$$

The inventory $I(t)$ is sampled with period T^* and a quantity $q = S - s$ is ordered when $I(t_i) < s$.

Multi-site inventory systems

If an inventory systems serves a demand d we have the following EOQ:

$$q^{(1)} = \sqrt{\frac{2kd}{h}}.$$

If an inventory systems is made of n identical sites (e.g. warehouses) and each site serves the same fraction of demand, i.e. d/n, then the EOQ for each site is

$$q^* = \sqrt{\frac{2kd}{nh}}$$

Hence the overall amount ordered in each period T in the multi-site system is

$$q^{(n)} = n\,q^* = n\,\sqrt{\frac{2kd}{nh}} = \sqrt{n}\,\sqrt{\frac{2kd}{h}} = \sqrt{n}\,q^{(1)}.$$

The relationship also holds for the average stocks:

$$\bar{I}^{(n)} = \sqrt{n}\,\bar{I}^{(1)}.$$