Optimal rotation of duties of hemodynamics units

A case study

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INFORMS Meeting 2014, San Francisco

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The context

The province of Milan has 17 hospitals with hemodynamics units for urgent treatment of patients affected by acute myocardial infarction.

Currently they are all on-duty during the nights and the week-ends.

Our goal is to evaluate an alternative scenario, based on a duty rotation of the hospitals.

We consider optimization from four different viewpoints:

- patients: timeliness of intervention;
- medical doctors: fair distribution of the duties;
- hospital administration: overall number of treated patients;
- regional administration: optimal configuration of hospitals.

Two scenarios

We consider nights and week-ends.

Scenario 1 (current):

- All hospitals are on duty.
- Patients are carried to the closest hospital.
- Physicians are available on call.
- Time bottleneck is the intervention time of the physician.

Scenario 2 (alternative):

- Some hospitals are on duty (rotation).
- Patients are carried to the closest hospital *on duty*.
- Physicians are already at the hospital.
- Time bottleneck is the transportation time of the patient.

Talk outline

Five problems:

1. minimization of the travel time from any point of the territory to the closest hemodynamics unit on duty in any time shift.
2. maximization of the minimum distance in time between two consecutive duties of a same unit.
3. study of the trade-off between the level of service and the balance in the assigned demand.
4. study of the effect of closing one or more hemodynamics units.
5. optimization of the transient from the current configuration with 17 units to a final configuration with fewer units.

For each problem I present integer linear programming formulations and the related computational results obtained with a common PC and a mathematical programming solvers (CPLEX 11).

Problem 1: optimization of the travel time

We are given:

- a set $\mathcal{I}$ of demand points in given positions (180 zones + 117 municipalities = 297 demand points);
- a set $\mathcal{J}$ of hospitals with hemodynamics units in given positions (17 units);
- a set of weekly time periods to be assigned (7 nights + 5 week-end shifts = 12 periods).
- **Parameter:** a number K of duties for each time period ($K = 17 \rightarrow$ Scenario 1).

From a GIS we computed d_{ij}^{max} and d_{ij}^{avg} $\forall i \in \mathcal{I}, \forall j \in \mathcal{J}$.
We could associate a population w_i to each demand point $i \in \mathcal{I}$.
The total population of the territory is $W = 2,920,444$ inhabitants.

Variables

We search for a cyclic schedule on a planning horizon of $\mathcal{T}$ periods equal to the number of hospitals.

$|\mathcal{T}| = 17$ and the number of periods in a week is 12: they are relatively prime. A fair rotation is obtained: the schedule repeats in the same days of the week every 17 weeks.

All units are on duty with the same average frequency in each of the 12 periods of the week.

Binary variables f_{ijt} indicate whether demand point $i \in \mathcal{I}$ is assigned to unit $j \in \mathcal{J}$ in period $t \in \mathcal{T}$.
Binary variables r_{jt} indicate whether unit $j \in \mathcal{J}$ is on duty in period $t \in \mathcal{T}$.

Objectives

We consider two objective functions representing equity and efficiency respectively:

- z_1^{max} is the maximum distance between any demand point in $\mathcal{I}$ and the closest unit on duty in any period;
- z_1^{avg} is the weighted average distance between any demand point in $\mathcal{I}$ and the closest unit on duty.

Obviously only one of two objective functions can be optimized at any time.

Their optimal values depend on K: the optimization is repeated for each value of $K = 1, \dots, 17$.

A compact formulation

The problem can be formulated as an assignment problem as follows.

$$\text{minimize } z_1^{max} = \max_{i\in\mathcal{I},j\in\mathcal{J},t\in\mathcal{T}}\{d_{ij}^{max} f_{ijt}\}$$

$$\text{minimize } z_1^{avg} = \frac{1}{|\mathcal{T}|}\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\frac{w_i}{W}d_{ij}^{avg} f_{ijt}$$

$$\begin{aligned}
\text{s.t. } & f_{ijt} \le r_{jt} && \forall i\in\mathcal{I}, \forall j\in\mathcal{J}, \forall t\in\mathcal{T}\\
& \sum_{t\in\mathcal{T}} r_{jt} = K && \forall j\in\mathcal{J}\\
& \sum_{j\in\mathcal{J}} r_{jt} = K && \forall t\in\mathcal{T}\\
& \sum_{j\in\mathcal{J}} f_{ijt} = 1 && \forall i\in\mathcal{I}, \forall t\in\mathcal{T}\\
& f_{ijt} \ge 0 && \forall i\in\mathcal{I}, \forall j\in\mathcal{J}, \forall t\in\mathcal{T}\\
& r_{jt} \text{ binary} && \forall j\in\mathcal{J}, \forall t\in\mathcal{T}.
\end{aligned}$$

A compact formulation

The model is similar to the Location Partitioning Problem with Balanced Cardinality Shifts (G. Andreatta, L. De Giovanni, P. Serafini, *Optimal shift partitioning of pharmacies*, AIRO conference 2012): *NP*-hardness results + preliminary computational tests with both heuristics and branch-and-price methods for determining cyclic schedules of night duties for pharmacies.

The compact model suffers from a very high degree of symmetry: all solutions obtained from one another by a permutation of the periods are equivalent.

The instance of our case study is too large (because of the number of demand points) to be solved by state-of-the-art integer linear programming solvers.

We devised a second model, based on an extended formulation.

An extended formulation

We consider the set $\mathcal{S}$ of subsets of $\mathcal{J}$ with cardinality K: each subset $S \in \mathcal{S}$ represents the units that are on duty simultaneously.

The number of subsets of cardinality K is combinatorial: $\binom{|\mathcal{J}|}{K}$.

With each subset $S \in \mathcal{S}$ we associate a cost:

$$c_S^{max} = \max_{i \in \mathcal{I}} \{ \min_{j \in S} \{ d_{ij}^{max} \} \}$$

$$c_S^{avg} = \sum_{i \in \mathcal{I}} \frac{w_i}{W} \min_{j \in S} \{ d_{ij}^{avg} \}$$

Explicit enumeration and evaluation of the subsets $\forall K = 1, \ldots, |\mathcal{J}|$: 17 problem instances of different size.

Min-max objective function

$$
\begin{aligned}
\text{minimize } & z_1^{max} = \max_{S \in \mathcal{S}} \{c_S^{max} y_S\} & \\
\text{s.t. } & \sum_{S \in \mathcal{S}} a_{jS} x_S = K & \forall j \in \mathcal{J} \\
& x_S \leq K y_S & \forall S \in \mathcal{S} \\
& y_S \text{ binary} & \forall S \in \mathcal{S} \\
& x_S \geq 0 \text{ and integer} & \forall S \in \mathcal{S}
\end{aligned}
$$

$y_S \in \{0, 1\}$: subset S belongs to the solution.
$x_S \in \mathcal{Z}_+$: how many times subset S is used in the solution.
a_{jS}: unit j appears in subset S.

Linearization

We can get rid of the min-max objective function, as follows.

$$
\begin{aligned}
\text{minimize } z_1^{max} &= z \\
\text{s.t. } z &\geq c_S^{max} y_S && \forall S \in \mathcal{S} \\
\sum_{S \in \mathcal{S}} a_{jS} x_S &= K && \forall j \in \mathcal{J} \\
x_S &\leq K y_S && \forall S \in \mathcal{S} \\
y_S &\text{ binary} && \forall S \in \mathcal{S} \\
x_S &\geq 0 \text{ and integer} && \forall S \in \mathcal{S}.
\end{aligned}
$$

Hierarchical min-max objective functions.

The optimal value of the min-max extended linearized model only depends on the selected subset with the largest cost. In other words the model is insensitive to all decisions concerning the use of subsets different from the worst one; hence it admits an exponentially large number of equivalent optimal solutions.

We consider a sequence of hierarchical objective functions: we solve a sequence of sub-problems aimed at minimizing the worst case, the second worst case, the third worst case and so on.

(See paper version for the details on the algorithm.)

Min-avg objective function

When the objective function z_1^{avg} is optimized, y variables are not needed.

$$\begin{aligned} \text{minimize } z_1^{avg} &= \frac{1}{|\mathcal{T}|} \sum_{S \in \mathcal{S}} c_S^{avg} x_S \\ \text{s.t.} \sum_{S \in \mathcal{S}} a_{jS} x_S &= K && \forall j \in \mathcal{J} \\ x_S &\geq 0 \text{ and integer} && \forall S \in \mathcal{S}. \end{aligned}$$

This min-avg extended formulation can be solved directly by an integer linear programming solver.

Computational results: min-max

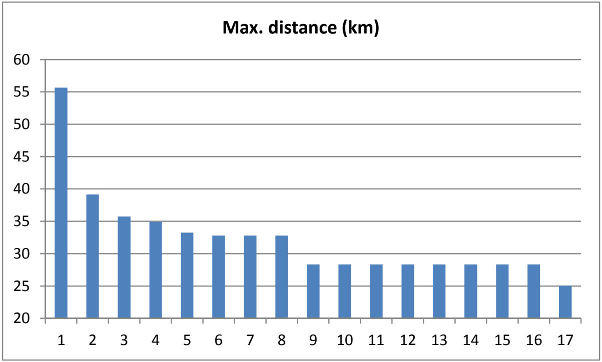

Figure: The optimal level of service z_1^{max*} as a function of the number K of units simultaneously on duty.

Improvements with K up to 6. No improvement for $K = 6, \ldots, 8$.
For $K = 9$ improvement by about 13.6%. No improvements for $K > 9$.

Computational results: min-avg

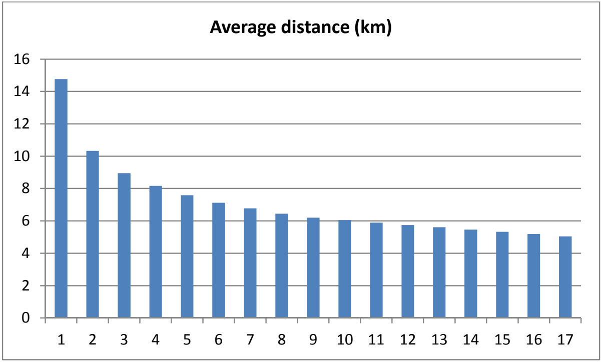

Figure: The optimal average distance z_1^{avg*} as a function of the number K of units simultaneously on duty.

Multi-objective optimization

z_1^{max} and z_1^{avg} are conflicting: Pareto-optimal region.

K	Min.max.dist. z_1^{max*}	Min.max.dist. z_1^{avg}	Min.max.dist. time	Min.avg.dist. z_1^{max}	Min.avg.dist. z_1^{avg*}	Min.avg.dist. time
1	55.656	14.764	0.25	55.656	14.764	-
2	39.148	10.869	0.45	39.655	10.327	0.02
3	35.747	9.331	1.14	35.747	8.948	0.04
4	34.897	8.540	4.72	35.747	8.169	0.03
5	33.243	7.955	10.33	33.243	7.588	0.16
6	32.806	7.557	19.32	32.806	7.114	0.14
7	32.806	7.060	39.16	32.806	6.771	0.26
8	32.806	6.761	82.91	32.806	6.448	0.35
9	28.335	6.439	117.05	28.335	6.204	0.39
10	28.335	6.214	67.59	28.335	6.045	0.60
11	28.335	5.982	53.66	28.335	5.892	0.22
12	28.335	5.802	22.80	28.335	5.747	0.18
13	28.335	5.622	8.90	28.335	5.607	0.10
14	28.335	5.500	1.87	28.335	5.466	0.03
15	28.335	5.329	0.49	28.335	5.326	0.02
16	28.335	5.185	0.33	28.335	5.185	0.02
17	25.034	5.045	0.29	25.034	5.045	-

Table: Values of z_1^{max} and z_1^{avg} with the two objective functions.

For almost all values of K the value of z_1^{max} obtained when optimizing z_1^{avg} is the optimal one. On the contrary, the optimal values of z_1^{avg} are usually missed when z_1^{max} is optimized.

Comparison between Scenario 1 and Scenario 2

Scenario 1: the waiting time for the physician is about 40 minutes.

Scenario 1 is a special case of Scenario 2 when $K = 17$:

- maximum distance for an ambulance is about 25 Kilometers;
- average distance for an ambulance is about 5 Kilometers.

If we assume an average speed of 50 km/h in the night on extra-urban roads, then 40 minutes are equivalent to 33.3 Kilometers.

- Min-max: Scenario 2 is better than Scenario 1 for $K \geq 5$.
- Min-avg: Scenario 2 is better than Scenario 1 for any value of K.

Problem 2: Optimization of the schedule

Critical aspects of Scenario 2:

- the frequency with which the units are on duty (problem 2);
- the frequency with which they receive patients (problem 3).

The former one is critical for the medical personnel: intense work duties in nights and week-ends are especially demanding.

Once Problem 1 has been optimized and an optimal multi-set $\mathcal{X}$ of unit subsets has been selected, they must be sequenced so that the duties for each unit be as far as possible in time from one another.

Problem formulation

The optimal solution of Problem 1 is a multi-set $\mathcal{X}$ of unit subsets.

Problem 2: find a permutation π of the elements of $\mathcal{X}$ that maximizes the minimum distance in time between any pair of subsets including the same unit.

The distance in time between two subsets is measured as the number of positions between them in the permutation.

This distance is computed modulo $|\mathcal{J}|$ because the schedule must be repeated cyclically.

maximize $z_2 = \min_{j \in \mathcal{J}} \{ \min_{S', S'' \in \mathcal{X} : j \in S' \cap S'', S' \neq S''} \{ (\pi(S') - \pi(S'')) \; mod \; |\mathcal{J}| \} \}$,

where S' and S'' are distinct items in the multi-set $\mathcal{X}$, possibly corresponding to the same subset of units.

The solution method

z_2 is a max-min objective function with a very small range of possible values.

A tentative feasible value z_2^{LB} is identified and a feasibility problem is solved imposing $z_2 \geq z_2^{LB} + 1$.

If a feasible solution is found, then $z_2^{LB} := z_2^{LB} + 1$ and repeat; otherwise stop: z_2^{LB} is optimal.

An ILP model

The feasibility problem is the following ILP:

$$\sum_{S \in \mathcal{X}} \rho_{pS} = 1 \qquad \forall p = 1, \ldots, |\mathcal{J}|$$

$$\sum_{p=1,\ldots,|\mathcal{J}|} \rho_{pS} = 1 \qquad \forall S \in \mathcal{X}$$

$$\sum_{q \in Q_p} \sum_{S \in X} a_{jS} \rho_{qS} \leq 1 \qquad \forall p = 1, \ldots, |\mathcal{J}|, \forall j \in \mathcal{J}$$

$$\rho_{pS} \text{ binary} \qquad \forall p = 1, \ldots, |\mathcal{J}|, \forall S \in \mathcal{X}$$

- $\rho_{pS} \in \{0, 1\}$: assignment of subset $S \in \mathcal{X}$ to position p in π;
- $Q_p = [p, \ldots, \min\{|\mathcal{J}|, p + z_2^{LB}\}] \cup [1..(p + z_2^{LB} - |\mathcal{J}|)]$.

An ILP model

To avoid solving infeasible instances, we solve this ILP which is always feasible.

$$\begin{aligned}
\text{maximize } \beta = \max\{ \sum_{p=1,\ldots,|\mathcal{J}|} \sum_{S \in \mathcal{X}} \rho_{pS} \} & \\
\text{s.t. } \sum_{S \in \mathcal{X}} \rho_{pS} \leq 1 & \quad \forall p = 1, \ldots, |\mathcal{J}| \\
\sum_{p=1,\ldots,|\mathcal{J}|} \rho_{pS} \leq 1 & \quad \forall S \in \mathcal{X} \\
\sum_{q \in Q_p} \sum_{S \in X} a_{jS} \rho_{qS} \leq 1 & \quad \forall p = 1, \ldots, |\mathcal{J}|, \forall j \in \mathcal{J} \\
\rho_{pS} \in \{0, 1\} & \quad \forall p = 1, \ldots, |\mathcal{J}|, \forall S \in \mathcal{X}
\end{aligned}$$

If $\beta^* < |\mathcal{J}|$, then the previous model is infeasible.

Computational results

The problem is trivial:

- for $K = 1$ ($z_2^* = |\mathcal{J}|$)
- for $K \geq |\mathcal{J}|/2$ ($z_2^* = 1$).

	Min.max.dist.		Min.avg.dist.	
K	z_2^*	time	z_2^*	time
2	7	1.20	7	0.94
3	3	0.19	4	0.71
4	2	148.59	2	3.83
5	1	9.39	2	15.10
6	1	1.67	1	0.05
7	1	0.03	1	0.06
8	1	0.05	1	0.05

Table: Minimum distance z_2^* between duties for a same unit.

The optimization of the min-avg objective function provides slightly better results.

Problem 3: Optimization of the budget balance

In Scenario 1 each hospital $j \in \mathcal{J}$ receives and treats a different average number of patients per unit of time. Since hospitals receive funds in proportion to the number of treatments they provide, it is important for them not to lose patients in the transition from Scenario 1 to Scenario 2.

Problem 3 has the same objectives and constraints as before, but in addition we also measure the average fraction of population that would be assigned to each unit in Scenario 2 and we impose that such a fraction be similar to the current one.

Variables

We use flow variables f_{ijt} with three indices:

- $i \in \mathcal{I}$ indicates the town/zone (origin);
- $j \in \mathcal{J}$ indicates the hemodynamics unit (destination);
- $t \in \mathcal{T}$ indicates the time period.

Assumption 1. All patients from the same *town* must be allocated to the same unit.

Assumption 2. Patients from the same *metropolitan area zone* can be allocated to different units.

Assumption 3. The assignment of demand points to the same unit can be different in different periods even when the same unit subset is on-duty.

Constraints

The subsets $\mathcal{X}$ obtained from the minimization of z_1^{avg} (Problem 1) are fixed.
We assign each selected subset to a time period $t \in \mathcal{T}$ (Problem 2):
$r_{jt} = 1$ iff unit $j \in \mathcal{J}$ belongs to the subset assigned to period $t \in \mathcal{T}$.

We only allow for the redistribution of patients in each time period.

We also impose that the maximum distance must not exceed the value $\overline{D}$ of z_1^{max} obtained from the optimization of z_1^{avg}.

We impose that the overall demand assigned to each unit $j \in \mathcal{J}$ be close to λ_j, which is the current “market share” of the unit; the overall absolute deviation is measured by a variable δ_j for each unit $j \in \mathcal{J}$.

A mixed-integer flow formulation

$$\text{minimize } z_3^{avg} = \frac{1}{|\mathcal{T}|} \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \frac{w_i}{W} d_{ij}^{avg} f_{ijt}$$

$$\text{minimize } z_3^{dev} = \max_{j\in\mathcal{J}} \{\frac{\delta_j}{\lambda_j}\}$$

$$\text{s.t. } \sum_{j\in\mathcal{J}} f_{ijt} = 1 \qquad \forall i \in \mathcal{I}\ \forall t \in \mathcal{T}$$

$$f_{ijt} \leq r_{jt} \qquad \forall i \in \mathcal{I}\ \forall j \in \mathcal{J}\ \forall t \in \mathcal{T}$$

$$f_{ijt} = 0 \qquad \forall t \in \mathcal{T}\ \forall i \in \mathcal{I}\ \forall j \in \mathcal{J} : d_{ij}^{max} > \overline{D}$$

$$\sum_{i\in\mathcal{I}} \sum_{t\in\mathcal{T}} w_i f_{ijt} \geq \lambda_j - \delta_j \qquad \forall j \in \mathcal{J}$$

$$\sum_{i\in\mathcal{I}} \sum_{t\in\mathcal{T}} w_i f_{ijt} \leq \lambda_j + \delta_j \qquad \forall j \in \mathcal{J}$$

$$f_{ijt} \text{ binary} \qquad \forall i \in \mathcal{I}^{towns}\ \forall j \in \mathcal{J}\ \forall t \in \mathcal{T}$$

$$f_{ijt} \geq 0 \qquad \forall i \in \mathcal{I}^{zones}\ \forall j \in \mathcal{J}\ \forall t \in \mathcal{T}$$

$$\delta_j \geq 0 \qquad \forall j \in \mathcal{J}.$$

Linearization

Objective function z_3^{dev} is linearized, introducing an auxiliary variable ϕ as follows:

$$\text{minimize } z_3^{dev} = \phi$$

$$\text{s.t. } \phi >= \frac{\delta_j}{\lambda_j} \ \ \forall j \in \mathcal{J}.$$

Patient flows are no longer constrained to comply with the minimum distance criterion. Hence the same subsets selected in Problem 1 may now correspond to a different value of the objective functions based on the distances.

Multi-objective optimization approach.

We consider the two extreme Pareto-optimal solutions:

- Distance first: optimization of z_3^{avg}; evaluation of z_3^{dev}.
- Balance first: optimization of z_3^{dev}; optimization of z_3^{avg} after fixing ϕ to its optimal value.

Computational results

	Distance first		Balance first	
K	z_3^{avg*}	z_3^{dev}	z_3^{avg}	z_3^{dev*}
1	14.764	**7.30**	14.764	7.30
2	10.327	**5.36**	12.029	0.39
3	8.948	**5.68**	10.101	0.69
4	8.169	**6.95**	10.508	0.07
5	7.588	**7.83**	10.150	0.04
6	7.115	**7.68**	8.909	0.00
7	6.772	**7.34**	8.111	0.00
8	6.448	**7.36**	7.629	0.00
9	6.204	**6.21**	7.243	0.00
10	6.045	**5.95**	6.857	0.00
11	5.892	**5.66**	6.571	0.00
12	5.747	**5.33**	6.313	0.00
13	5.607	**4.94**	6.071	0.00
14	5.466	**4.55**	5.848	0.00
15	5.326	**4.16**	5.646	0.00
16	5.185	**3.77**	5.462	0.00
17	5.052	**3.38**	5.322	0.00

Table: Results of multi-objective optimization of z_3^{avg} and z_3^{dev}.

The analysis can be repeated ($\forall K$) constraining the worst allowed value for any of the two objective functions, in order to compute Pareto-optimal solutions that are intermediate between these two.

Problem 4: Optimal selection-rotation

We assume the number and the location of hemodynamics units in the territory to be a variable.

The aim is to provide some insight into the trade-off between the number of hemodynamics units and their accessibility in Scenario 2.

We study how the optimal solutions of the optimization problems investigated so far (especially Problem 1) depend on the overall number and location of the units.

We use H to indicate the number of units available and the number of periods of the cyclic schedule.

The models are similar to those of Problem 1 with additional binary variables v indicating which units are active. Each model is solved for each value of $K = 1, \dots, H$.

Formulation: Min max

$$
\begin{aligned}
\text{minimize } & z_4^{max} = z & \\
\text{s.t. } & z \geq c_g^{max}\gamma_g & \forall g \in \mathcal{G} \\
& \sum_{g\in\mathcal{G}, S\in\mathcal{S}_g} a_{jS}x_S = Kv_j & \forall j \in \mathcal{J} \\
& \sum_{S\in\mathcal{S}_g} x_S \leq K\gamma_g & \forall g \in \mathcal{G} \\
& \sum_{j\in\mathcal{J}} v_j = H & \\
& \gamma_g \text{ binary} & \forall g \in \mathcal{G} \\
& v_j \text{ binary} & \forall j \in \mathcal{J} \\
& x_S \geq 0 \text{ and integer} & \forall S \in \mathcal{S}.
\end{aligned}
$$

Formulation: Min average

$$
\begin{aligned}
\text{minimize } z_4^{avg} &= \frac{1}{H} \sum_{S \in \mathcal{S}} c_S^{avg} x_S \\
\text{s.t.} \sum_{S \in \mathcal{S}} a_{jS} x_S &= K v_j && \forall j \in \mathcal{J} \\
\sum_{j \in \mathcal{J}} v_j &= H \\
v_j &\text{ binary} && \forall j \in \mathcal{J} \\
x_S &\geq 0 \text{ and integer} && \forall S \in \mathcal{S}.
\end{aligned}
$$

Computational results

K , H	7	8	9	10	11	12	13	14	15	16	17
1	39.655	40.270	41.390	41.933	43.228	44.532	44.532	46.097	46.434	47.760	55.656
2	34.897	33.559	34.897	34.897	35.747	35.747	35.747	38.837	38.987	38.987	39.148
3	32.806	32.806	32.806	33.243	33.243	33.243	33.243	34.897	34.897	35.747	35.747
4	28.335	**28.335**	32.806	32.806	32.806	32.806	32.806	33.243	33.243	33.243	34.897
5	28.335	28.335	28.335	28.335	32.806	32.806	32.806	32.806	32.806	33.243	33.243
6	28.335	28.335	28.335	28.335	28.335	28.335	28.335	32.806	32.806	32.806	32.806
7	25.034	28.335	28.335	28.335	28.335	28.335	32.806	32.806	32.806	32.806	32.806
8		28.335	28.335	28.335	28.335	28.335	28.335	28.335	28.335	28.335	**32.806**
9			28.335	28.335	28.335	28.335	28.335	28.335	28.335	28.335	28.335
10				28.335	28.335	28.335	28.335	28.335	28.335	28.335	28.335
11					28.335	28.335	28.335	28.335	28.335	28.335	28.335
12						25.034	28.335	28.335	28.335	28.335	28.335
13							28.335	28.335	28.335	28.335	28.335
14								28.335	28.335	28.335	28.335
15									28.335	28.335	28.335
16										28.335	28.335
17											25.034

Table: Optimization of z_4^{max} for different values of K (rows) and H (columns).

Computational results

K, H	7	8	9	10	11	12	13	14	15	16	17
1	11.929	12.124	12.278	12.499	12.692	12.888	13.078	13.372	13.677	13.995	14.764
2	9.706	9.641	9.762	9.721	9.827	9.808	9.914	9.988	10.101	10.209	10.327
3	8.577	8.624	8.602	8.682	8.725	8.728	8.795	8.823	8.841	8.885	8.948
4	7.779	7.714	7.829	7.888	7.930	7.940	8.017	8.054	8.100	8.129	8.169
5	7.384	7.332	7.270	7.204	7.309	7.372	7.423	7.461	7.496	7.544	7.588
6	6.830	6.920	6.878	6.861	6.834	6.789	6.901	6.973	7.018	7.068	7.114
7	6.203	6.421	6.502	6.507	6.524	6.525	6.531	6.515	6.619	6.691	6.771
8		5.853	6.700	6.170	6.231	6.264	6.293	6.311	6.320	6.336	6.448
9			5.659	5.829	5.944	6.016	6.063	6.107	6.121	6.155	6.204
10				5.482	5.655	5.766	5.840	5.912	5.944	5.988	6.045
11					5.365	5.515	5.617	5.718	5.778	5.830	5.892
12						5.263	5.395	5.523	5.612	5.679	5.747
13							5.172	5.329	5.446	5.527	5.607
14								5.132	5.271	5.376	5.466
15									5.095	5.225	5.326
16										5.066	5.185
17											5.045

Table: Optimization of z_4^{avg} for different values of K (rows) and H (columns).

Computational results

With either objective the level of service depends more on K than on H and the optimal value is almost monotonic with K for fixed H and with H for fixed K.

This suggests that H can be reduced without negatively affecting the patients, provided that K is suitably chosen and the cyclic schedule of duties is optimized.

For instance the maximum distance with $H = 8$ and $K = 4$ is smaller than that for $H = 17$ and $K = 8$ (which is counter-intuitive), provided that the active units and their rotation are chosen optimally.

Problem 5: Optimization of the transient

Assuming that a target configuration of H hemodynamics units has been identified, it is expected that it would take a rather long period of time to actually reach it; therefore the optimization of the transient is also important.

To optimize the transient we assume that $|\mathcal{J}| - H$ hemodynamics units are closed one at a time: this means that the transient is made of $|\mathcal{J}| - H + 1$ configurations.

The first one with $|\mathcal{J}|$ active units and the last one with H active units are given. We want to optimize the sequence of the $C = |\mathcal{J}| - H - 1$ intermediate configurations, leading from the former to the latter.

Variables

$\mathcal{J}^{off}$: subset of units to be sequentially closed.

Assumption. The number of units in each duty, K^c, has been suitably chosen for each intermediate configuration $c = 1, \ldots, C$.

- Binary variables v indicate whether each unit $j \in \mathcal{J}$ is active in each configuration $c = 1, \ldots, C$.
- Integer variables x indicate how many times pattern $S \in \mathcal{S}$ is used in configuration $c = 1, \ldots, C$.

An ILP model

$$\text{minimize } z_5^{avg} = \frac{1}{C}\sum_{c=1}^{C}\frac{1}{|\mathcal{J}|-c}\sum_{S\in\mathcal{S}} c_S^{avg} x_S^c$$

$$\text{s.t.} \sum_{S\in\mathcal{S}} a_{jS}\, x_S^c = K^c v_j^c \qquad \forall j\in\mathcal{J}\ \forall c = 1,\dots,C$$

$$\sum_{j\in\mathcal{J}} v_j^c = |\mathcal{J}| - c \qquad \forall c = 1,\dots,C$$

$$v_j^{c+1} \le v_j^c \qquad \forall j\in\mathcal{J}\ \forall c = 1,\dots,C-1$$

$$v_j^c = 1 \qquad \forall j \notin \mathcal{J}^{off}\ \forall c = 1,\dots,C$$

$$v_j^c \text{ binary} \qquad \forall j\in\mathcal{J}\ \forall c = 1,\dots,C$$

$$x_S^c \ge 0 \text{ and integer} \qquad \forall S\in\mathcal{S}\ \forall c = 1,\dots,C.$$

We solved this model with a min-avg objective function: the same can be done with a min-max objective.
An additional constraint can be added to forbid distances larger than a predefined threshold.

Computational results

Target configuration with $H = 10$ units (instead of 17) and duties with $K^{10} = 5$ units.

We set $K^{16} = K^{15} = K^{14} = K^{13} = 8$, $K^{12} = 7$ and $K^{11} = 6$.

The target configuration implies units 1, 2, 4, 6, 7, 15 and 17 to be turned off.

A threshold of 32.806 Kilometers is imposed on the maximum distance.

The model was solved in 35 seconds.

An optimal solution (1)

(H, K)	Hemodynamics units																	Avg.	Max.
x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	dist.	dist.
(16, 8)	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
8	1	0	0	1	1	0	0	1	0	1	0	1	0	1	1	0	0	6.351	29.926
8	0	0	1	0	0	1	1	0	1	0	1	0	1	0	0	1	1	6.321	27.490
(15, 8)	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1		
1	0	0	1	0	0	0	0	1	1	1	1	1	1	0	0	1	0	6.079	27.490
6	1	0	0	1	1	0	1	0	0	1	0	1	0	1	1	0	0	6.401	28.335
1	0	0	1	0	0	0	0	1	1	0	1	1	1	0	0	1	1	6.351	27.490
1	0	0	1	0	0	0	0	1	1	0	1	0	1	0	1	1	1	6.155	27.262
1	0	0	1	0	0	0	0	1	1	0	1	0	1	1	0	1	1	6.097	27.490
1	0	0	1	0	0	0	1	1	1	0	1	0	1	0	0	1	1	6.320	27.490
1	0	0	1	1	0	0	0	1	1	0	1	0	1	0	0	1	1	6.295	27.490
1	1	0	1	0	0	0	0	1	1	0	1	0	1	0	0	1	1	6.402	27.490
1	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	6.322	27.262
1	0	0	0	1	1	0	0	1	1	1	0	0	0	1	0	1	1	6.373	29.112
(14, 8)	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0		
2	0	0	0	1	0	0	0	1	1	1	1	0	1	1	0	1	0	6.002	27.490
2	1	0	0	1	1	0	1	0	0	1	0	1	0	1	1	0	0	6.401	28.335
2	1	0	1	1	1	0	1	0	0	1	0	0	0	1	1	0	0	6.451	28.335
2	1	0	0	1	1	0	1	1	0	1	0	0	0	1	1	0	0	6.421	28.335
2	0	0	1	0	1	0	1	0	1	0	1	1	1	0	0	1	0	6.344	27.490
2	1	0	1	0	0	0	0	1	1	0	1	1	1	0	0	1	0	6.407	27.490
2	0	0	1	0	0	0	0	1	1	0	1	1	1	0	1	1	0	6.159	27.262

Table: Optimization of the transient: from 17 units to 14...

An optimal solution (2)

(H, K)	Hemodynamics units																	Avg.	Max.
x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	dist.	dist.
(13, 8)	0	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0		
2	0	0	0	1	0	0	0	1	1	1	1	0	1	1	0	1	0	6.002	27.490
1	0	0	1	1	1	0	1	0	0	1	1	0	1	0	1	0	0	6.355	27.262
1	0	0	1	1	1	0	1	0	0	1	0	0	0	1	1	1	0	6.333	28.335
3	0	0	0	1	1	0	1	1	0	1	0	1	0	1	1	0	0	6.474	28.335
1	0	0	1	1	1	0	1	0	1	1	0	0	0	1	1	0	0	6.590	28.335
2	0	0	1	0	1	0	1	0	1	0	1	1	1	0	0	1	0	6.344	27.490
2	0	0	1	0	0	0	0	1	1	0	1	1	1	0	1	1	0	6.159	27.262
1	0	0	1	0	0	0	0	1	1	0	1	1	1	1	0	1	0	6.102	27.490
(12, 7)	0	0	1	1	1	0	0	1	1	1	1	1	1	1	1	1	0		
2	0	0	0	1	1	0	0	1	0	1	0	0	1	1	1	0	0	6.293	26.378
1	0	0	1	1	0	0	0	0	1	1	1	0	1	0	0	1	0	6.543	27.490
1	0	0	1	1	1	0	0	0	1	1	0	0	1	0	0	1	0	6.550	27.490
2	0	0	0	1	1	0	0	1	0	1	0	1	0	1	1	0	0	6.620	29.926
1	0	0	0	1	1	0	0	1	0	1	1	0	0	1	1	0	0	6.676	29.926
2	0	0	1	0	0	0	0	1	1	0	1	1	1	0	0	1	0	6.429	27.490
2	0	0	1	0	0	0	0	0	1	0	1	1	0	1	1	1	0	6.695	29.112
1	0	0	1	0	1	0	0	0	1	0	1	1	1	0	0	1	0	6.630	27.490
(11, 6)	0	0	1	0	1	0	0	1	1	1	1	1	1	1	1	1	0		
1	0	0	1	0	1	0	0	1	0	1	0	0	1	0	1	0	0	6.833	27.262
1	0	0	1	0	0	0	0	0	1	1	1	0	1	0	0	1	0	6.809	27.490
4	0	0	0	0	1	0	0	1	0	1	0	1	0	1	1	0	0	6.830	29.926
1	0	0	1	0	0	0	0	0	1	0	1	1	1	0	0	1	0	6.782	27.490
1	0	0	0	0	0	0	0	1	1	0	1	1	0	1	0	1	0	6.836	29.112
1	0	0	1	0	0	0	0	0	1	0	1	0	1	1	0	1	0	6.849	27.490
1	0	0	1	0	0	0	0	0	1	0	1	0	1	0	1	1	0	6.931	27.262
1	0	0	1	0	1	0	0	0	1	0	1	0	1	0	0	1	0	7.049	27.490
Transient																		6.443	29.926

Table: ...and to 14 to 10.

Conclusions

The level of service can be improved by switching to Scenario 2 with about one third of the units on duty.

Costs can be reduced, because K operators on duty may cost less than $|\mathcal{J}|$ operators available-on-call.

Consecutive duties can be avoided only if about one third of the units are on duty simultaneously.

Keeping 4-6 units on duty is enough to ensure an “equitable” partition of the demand without harming the transportation time.

It is possible to reduce the number of units on the territory without worsening the transportation time.

The transient can also be optimized to make the transition smooth and to avoid time periods with unsatisfactory service level.

Future developments

Design and implementation of specialized mathematical programming algorithms to solve larger instances to optimality or within a small approximation factor.

Paper version:
F. Lazzaroni, G. Righini, submitted to *O.R. for Health Care*.