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ORIGINAL PAPER

The hierarchical traveling salesman problem

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Abstract The distribution of relief aid is a complex problem where the operations have to be managed efficiently due to limited resources. We present a routing problem for relief operations whose primary goal is to satisfy demand for relief supplies at many locations taking into account the urgency of each demand. We have a single vehicle of unlimited capacity. Each node (location) has a demand and a priority. The priority indicates the urgency of the demand. Typically, nodes with the highest priorities need to be visited before lower priority nodes. We describe a new and interesting model for humanitarian relief routing that we call the hierarchical traveling salesman problem (HTSP). We compare the HTSP and the classical TSP in terms of worst-case behavior. We obtain a simple, but elegant result that exhibits the fundamental tradeoff between efficiency (distance) and priority and we provide several related observations and theorems.

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1 Introduction

We present a model for humanitarian relief routing. Suppose, as a result of a natural disaster such as an earthquake, tsunami, or hurricane, there are demands at many locations for relief supplies such as food, bottled water, blankets, or medical packs. Some locations are in more urgent need of supplies than other locations. Demand locations and the depot are nodes and each node (other than the depot) has a priority for a single relief product. The priorities indicate the urgency of the demand at each location; priority 1 nodes are in most urgent need of service. To begin, we assume that priority 1 nodes must be served before priority 2 nodes, priority 2 nodes must be served before priority 3 nodes, and so on. Four scenarios in which this model might apply are shown in Fig. 1.

We assume that a single vehicle has enough capacity to satisfy the needs at all demand locations and that visits to nodes must strictly respect the node priorities (i.e., priority 1 nodes first, priority 2 nodes second, and so on). This defines a hierarchical traveling salesman problem (HTSP). With this definition in mind, we will explore the fundamental tradeoff between efficiency (distance) and priority.

There are several papers in the literature related to our work. Campbell et al. [2] discuss routing for relief efforts using two different objective functions. In the first, they minimize the latest arrival time at a node. In the second, they minimize the average arrival time. Ngueveu et al. [8] present the cumulative capacitated vehicle routing problem (CCVRP). In the CCVRP, the objective is to minimize the sum of arrival times at customers (instead of total length) subject to vehicle capacity constraints. This is similar to the second objective function discussed by Campbell et al. [2]. Balcik et al. [1] discuss last mile distribution in humanitarian relief chains. The authors consider two types of products over a planning horizon and develop routes by solving TSPs. Fiala Timlin and Pulleyblank [4] describe an operational problem involving helicopters that service an offshore oil field consisting of 45 drilling platforms. Each day, each helicopter flies a route over a subset of these platforms, performing pickups

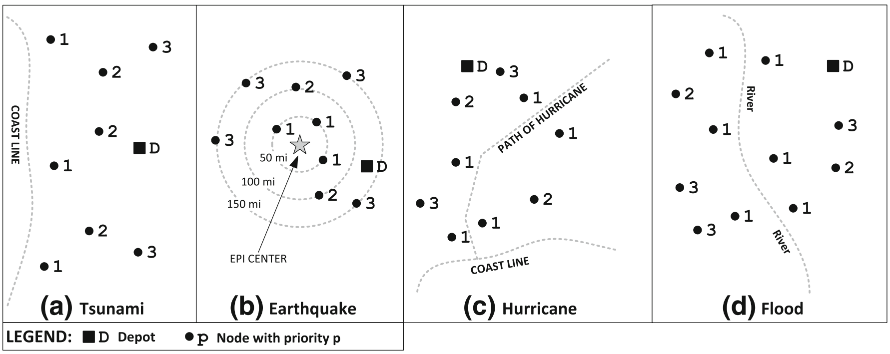

Fig. 1 Four scenarios for node priorities

and deliveries. Each platform has a priority and these are strictly respected. Although this helicopter routing problem (HRP) is very similar to the HTSP, there are several differences. In the HRP, there are pickups and deliveries and helicopter capacity is an important constraint. Another variant of the TSP related to the HTSP is the precedence constrained TSP (see Psaraftis [10]). In the clustered traveling salesman problem, the customers are clustered and all customers in a given cluster must be visited consecutively; there are no priorities (see Guttman-Beck et al. [5]).

A second potential application of the HTSP involves the routing of service technicians. Each day, your local gas and electric company routes service technicians to homes and businesses for minor repairs, major repairs, and new installations. Customers without heat (in the winter) or without air conditioning (in the summer) would be classified as priority 1 nodes. Other customers might be classified as priority 2 or priority 3 nodes. A similar application would arise for cable TV and internet providers.

A third potential (military) application of the HTSP involves the unmanned aerial vehicle (UAV) routing problem in which target priorities are important. Both static and dynamic versions of this application have received a lot of recent attention in the literature (e.g., see Yadlapalli et al. [11] and Chapter 5 of Mennell [7]).

2 Worst-case results

In solving the HTSP, we visit priority 1 nodes first, priority 2 nodes second, and so on. We pay a price for this in terms of travel distance. That is, if we ignore node priorities and solve the associated TSP, the resulting travel distance will be smaller. In the worst-case, how large a price do we pay? In this section, we address this question for the strict and relaxed versions of the HTSP.

Definition 1 For any positive integers p and q, we define $\left\lceil \frac{p}{q} \right\rceil$ to be the smallest integer which is greater than or equal to $\frac{p}{q}$. To be more specific, assume $p = kq + r$, where $0 \leq r < q$. Then, $\left\lceil \frac{p}{q} \right\rceil = k$ if $r = 0$ and $\left\lceil \frac{p}{q} \right\rceil = k + 1$ if $r > 0$.

Definition 2 The $d-$relaxed priority rule adds operational flexibility by allowing the vehicle to visit nodes of priority $\pi + 1, \ldots, \pi + d$ (if these priorities exist in the given instance) but not priority $\pi + d + l$ for $l \geq 1$ before visiting all nodes of priority π (for $\pi = 1, 2, \ldots, P$).

The impact of d on efficiency is illustrated in Fig. 2 with $P = 4$. In Fig. 2a, we have $d = 3$, which yields a TSP. In Fig. 2b, we have $d = 1$, which results in a much longer tour.

Theorem 1 *Let $Z^*_{d,P}$ and Z^*_{TSP} be the optimal tour length (distance) for the HTSP with the $d-$relaxed priority rule and for the TSP (without any priorities), respectively. Given that the triangle inequality holds, we have*

(a) $Z^*_{0,P} \leq P\ Z^*_{TSP}$, *and*
(b) $Z^*_{d,P} \leq \left\lceil \frac{P}{d+1} \right\rceil Z^*_{TSP}$.

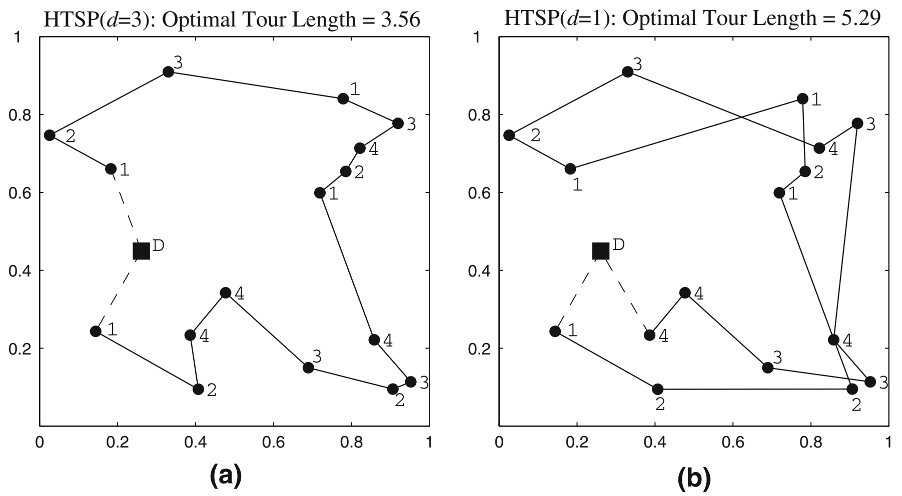

Fig. 2 Efficiency versus priority

Proof (a) Suppose we have an optimal TSP tour τ^* of length Z^*_{TSP} and, for now, assume that $P = 3$. Think of priority 1 nodes as red, priority 2 nodes as blue, and priority 3 nodes as green. Now, construct a tour $\tau(1)$ over the priority 1 nodes as follows. From the depot, visit these nodes in the same sequence as they appear in τ^*, and then return to the depot. Since the triangle inequality holds, length of $\tau(1) \leq Z^*_{TSP}$. Applying the same idea for priority 2 nodes and priority 3 nodes yields length of $\tau(2) \leq Z^*_{TSP}$ and length of $\tau(3) \leq Z^*_{TSP}$. Let $\tau(1) = (D, R_1, \ldots, R_r, D)$, $\tau(2) = (D, B_1, \ldots, B_b, D)$, and $\tau(3) = (D, G_1, \ldots, G_g, D)$. Then, $\tau = (D, R_1, \ldots, R_r, B_1, \ldots, B_b, G_1, \ldots, G_g, D)$ is a feasible solution to the HTSP with length of $\tau \leq$ length of $\tau(1)+$length of $\tau(2)+$ length of $\tau(3) \leq 3Z^*_{TSP}$. This again follows from the triangle inequality. For P priority classes, this logic yields: $Z^*_{0,P} \leq$ length of $\tau \leq P\ Z^*_{TSP}$.

(b) We can assume that $P = k(d+1) + r$, where $0 \leq r < d+1$ and k, d, and r are integers. Again, assume an optimal TSP tour τ^* of length Z^*_{TSP}. We divide all the nodes in the P priority classes into $\left\lceil \frac{P}{d+1} \right\rceil$ sets as follows. The first set, S_1, contains all nodes with priorities 1 to $d+1$. The second set, S_2, contains all nodes with priorities $d+2$ to $2(d+1)$, and so on. The kth set, S_k, contains all nodes with priorities $(k-1)(d+1)+1 = (k-1)d+k$ to $k(d+1)$. If $r > 0$, there is a $(k+1)th$ set, S_{k+1}, which contains all nodes with priorities $k(d+1)+1$ to $k(d+1)+r = P$. Tour $\tau(1)$ starts at the depot and visits all nodes in S_1 in the same sequence as they appear in τ^* and then it returns to the depot. By the triangle inequality, length of $\tau(1) \leq Z^*_{TSP}$. We handle sets $S_2, S_3, \ldots, S_{\left\lceil \frac{P}{d+1} \right\rceil}$ similarly and obtain tours $\tau(2), \tau(3), \ldots, \tau(\left\lceil \frac{P}{d+1} \right\rceil)$, where the length of $\tau(m) \leq Z^*_{TSP}$ for $1 \leq m \leq \left\lceil \frac{P}{d+1} \right\rceil$. We can connect these tours, as in part (a), to obtain a feasible

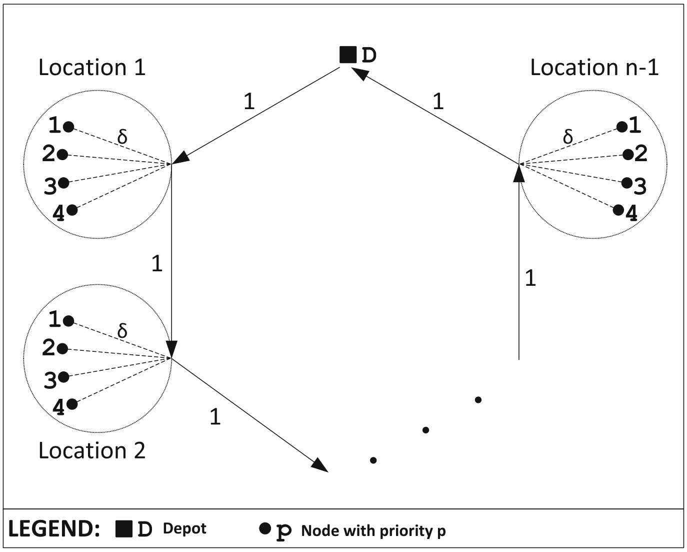

Fig. 3 Worst-case example

tour for the HTSP. By summing, length of $\tau \leq \left\lceil \frac{P}{d+1} \right\rceil Z^*_{TSP}$. Since $Z^*_{d,P}$ is the length of the optimal HTSP solution, we have $Z^*_{d,P} \leq \left\lceil \frac{P}{d+1} \right\rceil Z^*_{TSP}$. $\square$

We point out two special cases of part (b). First, if $d = 0$, we have part (a). Second, if $d = P - 1$, then $Z^*_{d,P} = Z^*_{TSP}$.

Next, we show that the bounds in Theorem 1, $\left(\text{namely, } P \text{ and } \left\lceil \frac{P}{d+1} \right\rceil\right)$ cannot be improved. Since part (a) is a special case of part (b), we focus on the bound $\left\lceil \frac{P}{d+1} \right\rceil$.

In Fig. 3, we have a polygon with n vertices. The depot (D) is the vertex at the top. Each other vertex represents a location (1 to $n - 1$) that contains P nodes; each has a different priority. The length of each edge of the polygon is 1. The lengths of the edges within each vertex (other than the depot) are $\delta << 1$. Since δ is so small, it can be omitted in the calculation of total tour length.

It is easy to observe that $Z^*_{TSP} = n$. The corresponding tour starts at the depot and goes through each of the $n - 1$ locations in counterclockwise order and then returns to the depot. The optimal tour of the HTSP can be constructed as follows. We start at the depot and go through each location in counterclockwise order. At each location, we visit all nodes with priorities 1 to $d + 1$. After we reach location $n - 1$, we finish visiting all nodes in S_1. Next, we can visit all nodes in location $n - 1$ with priorities $d + 2$ to $2(d + 1)$ and go through each location in clockwise order until we reach location 1.

Again, at each location, we visit all nodes with priorities from $d+2$ to $2(d+1)$. After we reach location 1, we finish visiting all nodes in S_2. Next, we visit all nodes with priority from $2(d+1)+1$ to $3(d+1)$ and go through all locations in counterclockwise order until we reach location $n-1$. We continue the procedure until all the nodes have been visited. At this point, we will be either at location 1 or location $n-1$. From here, we return to the depot.

Now, we can calculate the total length of the tour. We travel from location 1 to location $n-1$ (or the reverse) a total of $\left\lceil \frac{P}{d+1} \right\rceil$ times. We begin the tour by traveling from the depot to location 1 and we end the tour by traveling from either location 1 or location $n-1$ back to the depot. Therefore, we have $Z^*_{d,P} = \left\lceil \frac{P}{d+1} \right\rceil (n-2)+2$. Dividing by Z^*_{TSP}, we obtain

$$\frac{Z^*_{d,P}}{Z^*_{TSP}} = \frac{\left\lceil \frac{P}{d+1} \right\rceil (n-2)+2}{n} = \left\lceil \frac{P}{d+1} \right\rceil - \frac{2}{n}\left(\left\lceil \frac{P}{d+1} \right\rceil - 1\right), \tag{1}$$

which goes to $\left\lceil \frac{P}{d+1} \right\rceil$ as $n \to \infty$. Therefore, a tight, worst-case bound for $\frac{Z^*_{d,P}}{Z^*_{TSP}}$ is $\left\lceil \frac{P}{d+1} \right\rceil$.

Based on the proof of Theorem 1, we can make the following observations:

Observation 1 *If we divide the nodes in the P priority classes into $\left\lceil \frac{P}{d+1} \right\rceil$ sets, as indicated in part (b), we can solve (approximately) a TSP over each set using our favorite TSP heuristic. We can connect the resulting tours, as in part (a), to obtain a feasible tour for the HTSP.*

Observation 2 *Alternatively, we can solve (approximately) a TSP over the entire set of nodes using our favorite TSP heuristic and obtain a feasible tour for the HTSP by following the part (b) proof.*

Observation 3 *Suppose we select Christofides' heuristic (see Christofides* [3]*) in Observation 2. If we let $Z^c_{d,P}$ be the length of the resulting feasible solution to the HTSP, we have*

$$Z^c_{d,P} \le \frac{3}{2} \cdot \left\lceil \frac{P}{d+1} \right\rceil Z^*_{TSP}. \tag{2}$$

Observation 4 *The HTSP (with $d = 0$) can be modeled and solved as an asymmetric TSP.*

In Theorem 1, we presented a worst-case result for the HTSP on a graph. Next, we present a similar result (with a smaller bound) for the HTSP on the line.

Theorem 2 *Let $Z^*_{d,P}$ and Z^*_{TSP} be the optimal tour length (distance) for the HTSP on the line with the $d-$relaxed priority rule and for the TSP (without any priorities), respectively. Given that the triangle inequality holds, we have*

(a) $Z^*_{0,P} \le \frac{1}{2}(P+1)Z^*_{TSP}$, *and*
(b) $Z^*_{d,P} \le \frac{1}{2}\left(\left\lceil \frac{P}{d+1} \right\rceil + 1\right) Z^*_{TSP}$.

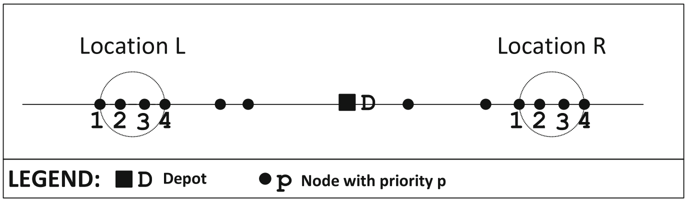

Fig. 4 The HTSP on the line

Discussion The proof for Theorem 2, using the configuration of nodes shown in Fig. 4, where L is the left-most point, D is the depot, and R is the right-most point, is easy to prove. The bound in (b) is tight. If one takes an example in which nodes of each priority class are located at both L and R, this becomes apparent. Here we assume the distance between any two nodes at L (and at R) is less than δ. Of course, (a) is a special case of (b).

3 A related result

Suppose we define the Hierarchical Chinese Postman Problem (HCPP) so that it is analogous to the HTSP. Here, each arc has a length and a priority. Arcs may be traversed in any order, but they must be serviced according to the $d-$relaxed priority rule (this definition is a generalization of the one presented by Korteweg and Volgenant [6]). The HCPP has obvious applications in snow removal over a road network. A result similar to Theorem 1 holds.

Theorem 3 *Let $W^*_{d,P}$ and W^*_{CPP} be the optimal cycle length (distance) for the HCPP with the $d-$relaxed priority rule and for the CPP (without any priorities), respectively. Given that the triangle inequality holds, we have*

(a) $W^*_{0,P} \leq P W^*_{CPP}$, *and*
(b) $W^*_{d,P} \leq \left\lceil \frac{P}{d+1} \right\rceil W^*_{CPP}$.

Discussion The proof of Theorem 1 can be easily applied to Theorem 3. Furthermore, we can show that the bounds in Theorem 3 are tight by slightly modifying Fig. 3. In particular, replace each of the $n-1$ locations in Fig. 3 with a cycle of length δ containing an arc from each priority class. Assume that the arcs forming the larger circle (each of length 1) are of priority 1.

4 Conclusions

In this paper, we introduced the HTSP which considers the priority of a location (node) with respect to humanitarian relief operations. We proposed a $d-$relaxed priority rule that provides flexibility to the decision maker in terms of capturing tradeoffs between total distance and node priorities. Furthermore, we derived worst-case bounds for the HTSP with respect to the TSP and were able to show that the bounds are tight.

We discussed other potential applications of the HTSP as well as related observations and results.

Our work could be extended to handle a fleet of capacitated vehicles, locations with demands for multiple products, or a longer planning horizon (e.g., a week). Furthermore, the node priorities might not be known with certainty or they may become known when a vehicle gets close enough to the node.

We point out that Panchamgam [9] formulated the HTSP and several extensions as mixed integer programs. HTSP instances with 30 or so nodes were solved to optimality using CPLEX.

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