Lagrangean relaxation with AMPL: examples

Operational Research Complements

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The CPMP

Capacitated p-median problem.

Data:

- a graph $G = (N, A)$;
- a demand $d_i \ \forall i \in N$;
- a supply $s_i \ \forall i \in N$;
- a unit transportation cost $c_{ij} \ \forall (i, j) \in A$;
- a number p of facilities to build.

Variables:

- Location: $y_i \in \{0, 1\} \ \forall i \in N$ represents the decision of building a facility in node $i \in N$;
- Allocation: $x_{ij} \geq 0$ indicates the amount supplied to node j from a facility located in node i.

Constraints:

- Only p facilities can be built;
- Supply can be provided only by the nodes where a facility is built;
- The demand of all nodes must be satisfied;
- The supply that each facility can provide is bounded.

Objective: Minimize total transportation cost.

Model A

$$
\begin{aligned}
\text{minimize } z = \sum_{i \in N, j \in N} c_{ij} x_{ij} & \\
\text{s.t. } \sum_{i \in N} x_{ij} \geq d_j & \qquad \forall j \in N \\
\sum_{j \in N} x_{ij} \leq s_i y_i & \qquad \forall i \in N \\
\sum_{i \in N} y_i \leq p & \\
y_i \in \{0, 1\} & \qquad \forall i \in N \\
x_{ij} \geq 0 & \qquad \forall (i, j) \in A
\end{aligned}
$$

Lagrangean relaxation A

$$
\begin{aligned}
\text{minimize } z_{LR}^A = & \sum_{i\in N, j\in N} c_{ij}x_{ij} + \sum_{j\in N} \lambda_j (d_j - \sum_{i\in N} x_{ij}) & \\
\text{s.t. } & \sum_{j\in N} x_{ij} \le s_i y_i & \forall i \in N \\
& \sum_{i\in N} y_i \le p & \\
& y_i \in \{0,1\} & \forall i \in N \\
& x_{ij} \ge 0 & \forall (i,j) \in A
\end{aligned}
$$

Lagrangean relaxation B

$$
\begin{aligned}
\text{minimize } z_{LR}^{B} = & \sum_{i\in N, j\in N} c_{ij}x_{ij} + \sum_{j\in N}\lambda_j(d_j - \sum_{i\in N} x_{ij}) & \\
\text{s.t. } & \sum_{j\in N} x_{ij} \le s_i y_i & \forall i \in N \\
& \sum_{i\in N} y_i \le p & \\
& y_i \in \{0,1\} & \forall i \in N \\
& 0 \le x_{ij} \le d_j & \forall (i,j) \in A
\end{aligned}
$$

Lagrangean relaxation C

$$
\begin{aligned}
\text{minimize } z_{LR}^{C} = \sum_{i\in N, j\in N} c_{ij}x_{ij} + \sum_{j\in N} \lambda_j \Big(d_j - \sum_{i\in N} x_{ij}\Big) & \\
\text{s.t. } \sum_{j\in N} x_{ij} \le s_i & \qquad \forall i \in N \\
\sum_{i\in N} y_i \le p & \\
x_{ij} \le d_j y_i & \qquad \forall (i,j) \in A \\
y_i \in \{0,1\} & \qquad \forall i \in N \\
x_{ij} \ge 0 & \qquad \forall (i,j) \in A
\end{aligned}
$$

The MMCF

Multi-commodity Min Cost Flow problem.

Data:

- a graph $G = (N, A)$;
- a set of commodities K;
- a supply or demand b_{ik} $\forall i \in N, k \in K$;
- a unit cost c_{ijk} $\forall (i, j) \in A, k \in K$;
- a total capacity u_{ij} $\forall (i, j) \in A$.

Variables:

- Flow: $x_{ijk} \geq 0$ indicates the amount of flow of commodity $k \in K$ or arc $(i, j) \in A$.

Constraints:

- Flow conservation constraints for each commodity.
- Capacity constraint on each arc.

Objective: Minimize total flow cost.

MMCF

$$\begin{aligned}
\text{minimize } z = \sum_{i \in N, j \in N} \sum_{k \in K} c_{ijk} x_{ijk} & \\
\text{s.t. } \sum_{i \in N} x_{ijk} + b_{jk} = \sum_{i \in N} x_{jik} & \qquad \forall j \in N, k \in K \\
\sum_{k \in K} x_{ijk} \leq u_{ij} & \qquad \forall (i, j) \in A \\
x_{ijk} \geq 0 & \qquad \forall (i, j) \in A, k \in K
\end{aligned}$$

Lagrangean relaxation

$$
\begin{aligned}
\text{minimize } z_{LR} = & \sum_{i \in N, j \in N} \sum_{k \in K} c_{ijk} x_{ijk} + \sum_{(i,j) \in A} \lambda_{ij} \left(\sum_{k \in K} x_{ijk} - u_{ij} \right) & \\
\text{s.t. } & \sum_{i \in N} x_{ijk} + b_{jk} = \sum_{i \in N} x_{jik} & \forall j \in N, k \in K \\
& x_{ijk} \geq 0 & \forall (i,j) \in A, k \in K
\end{aligned}
$$