Lagrangean relaxation - exercises

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1 Set covering

We start from the following Set Covering Problem instance:

$$\begin{aligned}
\min z = x_1 + 2x_2 + x_3 + 2x_4 + x_5 & & \\
x_1 + x_2 \qquad + x_4 \qquad & \geq & 1 \\
x_2 + x_3 \qquad\qquad & \geq & 1 \\
x_2 \qquad + x_4 + x_5 & \geq & 1 \\
x_3 + x_4 + x_5 & \geq & 1 \\
x_1, x_2, x_3, x_4, x_5 & \in & \{0,1\}
\end{aligned}$$

We relax all four constraints with Lagrangean relaxation and we obtain:

$$\begin{aligned}
\min z_{LR}(\lambda) =& x_1 + 2x_2 + x_3 + 2x_4 + x_5 + \\
&+ \lambda_1(1 - x_1 - x_2 - x_4) + \lambda_2(1 - x_2 - x_3) + \\
&+ \lambda_3(1 - x_2 - x_4 - x_5) + \lambda_4(1 - x_3 - x_4 - x_5) \\
& x \in \{0,1\}
\end{aligned}$$

which we can rewrite as

$$\begin{aligned}
\min z_{LR}(\lambda) =& (1 - \lambda_1)x_1 + (2 - \lambda_1 - \lambda_2 - \lambda_3)x_2 + (1 - \lambda 2 - \lambda_4)x_3 + \\
&+ (2 - \lambda_1 - \lambda_3 - \lambda_4)x_4 + (1 - \lambda_3 - \lambda 4)x_5 + \\
&+ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\
& x \in \{0,1\}
\end{aligned}$$

Since this is a Lagrangean relaxation, if we solve problem LR to optimality for any given choice of $\lambda \geq 0$ we always obtain a valid lower bound, i.e. $z^*_{LR}(\lambda) \leq z^*$.

If we use very low values for the multipliers, we may obtain infeasible solutions from LR, because constraint violations are not penalized enough. For instance with $\lambda = (0.2, 0.7, 0.7, 0, 4)$ we have

$$\begin{aligned}
\min z_{LR}(\lambda) =& 0.8x_1 + 0.4x_2 - 0.1x_3 + 0.7x_4 - 0.1x_5 + 2.0 \\
& x \in \{0,1\}
\end{aligned}$$

whose optimal solution is $x^* = (0, 0, 1, 0, 1)$ with $z^*_{LR}(\lambda) = 1.8$. With this solution row 1 remains uncovered.

If we use very large values for the multipliers, we are likely to obtain a feasible solution but a weak lower bound. For instance if we choose $\lambda = (10, 10, 10, 10)$ we have

$$\begin{aligned} \min z_{LR}(\lambda) = & - 9x_1 - 28x_2 - 19x_3 - 28x_4 - 19x_5 + 40 \\ & x \in \{0, 1\} \end{aligned}$$

whose optimal solution is $x^* = (1, 1, 1, 1, 1)$ with $z^*_{LR}(\lambda) = -63$. With this solution all rows are covered but the lower bound is useless.

1.1 Lagrangean heuristic

Starting from an infeasible solution obtained by Lagrangean relaxation, it is possible to repair it in a heuristic way, in order to obtain a feasible solution and hence an upper bound $z_{UB} \geq z^*$.

For instance, starting from $x^* = (0, 0, 1, 0, 1)$ which is optimal for $\lambda = (0.2, 0.7, 0.7, 0, 4)$, we can choose the minimum cost column that we need to insert in order to cover the uncovered rows. Here we have only one uncovered row, that is row 1. The minimum cost column covering row 1 is column 1. Hence we obtain $x = (1, 0, 1, 0, 1)$ which is now feasible. Its cost is equal to 3 and this is an upper bound (it is also the optimal value but so far we do not have obtained any guarantee of this).

A heuristic procedure like this is very fast and it can be repeated in every node of a branch-and-bound tree.

1.2 Subgradient optimization

Though being convex, function $f^*_{LR}(\lambda)$ does not have an analytical expression and it is not everywhere differentiable. The subgradient optimization method is an iterative procedure to search for its optimal value. Starting from the current optimal solution $x^*_(\lambda^{(k)})$, obtained at iteration k, it modifies the values of the Lagrangean multipliers so that:

- multipliers corresponding to violated constraints are increased;
- multipliers corresponding to active constraints are left unchanged;
- multipliers corresponding to respected and non active constraints are decreased.

This is equivalent to increase each multiplier by an amount proportional to the subgradient, i.e. to the amount by which the constraint is violated in $x^*(\lambda^{(k)})$: $s^{(k)} =$

$b - Ax^*(\lambda^{(k)})$. The updating formula to compute $\lambda^{(k+1)}$ at iteration $k+1$ from $\lambda^{(k)}$ at iteration k is as follows:

$$\lambda^{(k+1)} = \max\{\lambda^{(k)} + \frac{|z_{UB} - z^*_{LR}(\lambda^{(k)})|}{\left\|s^{(k)}\right\|^2} s^{(k)}, 0\}$$

where z_{UB} is an upper bound. If the gap between the current upper bound z_{UB} and the current lower bound $z^*_{LR}(\lambda^{(k)}$ is large, the step is also large; if the gap is small, the step is also small. When the value of a multiplier would become negative according to the update formula, it is set to 0 because Lagrangean relaxation of inequality constraints requires all multipliers to be non-negative.

Since we are using z_{UB} instead of z^* (which is obviously unknown), we are introducing an approximation which enlarges the step. Therefore the convergence of the method is no longer guaranteed. For this purpose a decreasing step coefficient t is used, to scale down the steps. A common heuristic rule is to start with $t^{(0)} = 2$ and to halve its value every 30 or so iterations. hence we are using the following update formula:

$$\lambda^{(k+1)} = \max\{\lambda^{(k)} + t^{(k)} \frac{|z_{UB} - z^*_{LR}(\lambda^{(k)})|}{\left\|s^{(k)}\right\|^2} s^{(k)}, 0\}$$

Subgradient optimization terminates in two cases:

- when we reach a good enough lower bound (i.e. $z^*_{LR}(\lambda^{(k)})$ is very close to z_{UB})
- when $t^{(k)}$ is so small that the multipliers do not change significantly from one iteration to the other.

Remark. When we are solving ILP instances with integer data we know that z^* is also integer. Hence we can round up $z^*_{LR}(\lambda^{(k)})$ in the end test (not in the multipliers update formula).

1.3 Solution of our example

We start with $\lambda^{(0)} = (0, 0, 0, 0)$ and $t^{(0)} = 1$.

Lagrangean subproblem: iteration 1. The relaxed problem is

$$\begin{aligned} \min z_{LR}(\lambda^{(0)}) = x_1 + 2x_2 + x_3 + 2x_4 + x_5 & \\ x_1, x_2, x_3, x_4, x_5 \quad \in \quad \{0, 1\} & \end{aligned}$$

The optimal solution is $x^*(\lambda^{(0)}) = (0, 0, 0, 0, 0)$ with $z^*_{LR}(\lambda^{(0)}) = 0$. All rows are uncovered.

Lagrangean heuristic. We iteratively look for the minimum cost column that covers an uncovered row.
Iteration 1: row 1. We select column 1 of cost 1. Rows 2, 3 and 4 remain uncovered.
Iteration 2: row 2. We select column 3 of cost 1. Row 3 remains uncovered.
Iteration 3: row 3. We select column 5 of cost 1. Now all rows are covered.
We have computed the heuristic solution $x = (1, 0, 1, 0, 1)$ with cost $z_{UB} = 3$.

Subgradient optimization: iteration 1. Using the update formula we compute:

$$\lambda^{(1)} = \max\{\lambda^{(0)} + t^{(0)} \frac{|z_{UB} - z^*_{LR}(\lambda^{(0)})|}{\|s^{(0)}\|^2} s^{(0)}, 0\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 1 \frac{|3-0|}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{bmatrix}$$

Lagrangean subproblem: iteration 2. With $\lambda^{(1)} = (0.75, 0.75, 0.75, 0.75)$ we have

$$\begin{aligned}
\min z_{LR}(\lambda^{(1)}) = x_1 + 2x_2 + x_3 + 2x_4 + x_5 \quad & + \\
\frac{3}{4}(1 - x_1 - x_2 \qquad - x_4 \qquad) \quad & + \\
\frac{3}{4}(1 \qquad - x_2 - x_3 \qquad\qquad) \quad & + \\
\frac{3}{4}(1 \qquad - x_2 \qquad - x_4 - x_5) \quad & + \\
\frac{3}{4}(1 \qquad\qquad - x_3 - x_4 - x_5) \quad & = \\
\frac{1}{4}x_1 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_4 - \frac{1}{2}x_5 + 3 & \\
x_1, x_2, x_3, x_4, x_5 \quad & \in \quad \{0, 1\}
\end{aligned}$$

The optimal solution is $x^*(\lambda^{(1)}) = (0, 1, 1, 1, 1)$ with $z^*_{LR}(\lambda^{(1)}) = 1.50$. In this solution all rows are covered.

Subgradient optimization: iteration 2. We have $s^{(1)} = (-1, -1, -2, -2)$. Using the update formula we compute:

$$\lambda^{(2)} = \begin{bmatrix} 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{bmatrix} + 1 \frac{|3 - 3/2|}{10} \begin{bmatrix} -1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 12/20 \\ 12/20 \\ 9/20 \\ 9/20 \end{bmatrix}$$

Lagrangean subproblem: iteration 3. With $\lambda^{(2)} = (0.6, 0.6, 0.45, 0.45)$ we have

$$\begin{aligned}
\min z_{LR}(\lambda^{(2)}) = x_1 + 2x_2 + x_3 + 2x_4 + x_5 \quad & + \\
\frac{12}{20}(1 - x_1 - x_2 \qquad - x_4 \qquad) \quad & + \\
\frac{12}{20}(1 \qquad - x_2 - x_3 \qquad\qquad) \quad & + \\
\frac{9}{20}(1 \qquad - x_2 \qquad - x_4 - x_5) \quad & + \\
\frac{9}{20}(1 \qquad\qquad - x_3 - x_4 - x_5) \quad & = \\
0.40x_1 + 0.35x_2 - 0.05x_3 + 0.50x_4 + 0.10x_5 + 2.1 & \\
x_1, x_2, x_3, x_4, x_5 \quad & \in \quad \{0, 1\}
\end{aligned}$$

The optimal solution is $x^*(\lambda^{(2)}) = (0, 0, 1, 0, 0)$ with $z^*_{LR}(\lambda^{(2)}) = 2.05$. In this solution rows 1 and 3 are uncovered.

We can observe that the lower bound $z^*_{LR}(\lambda^{(2)}) = 2.05$ can be rounded up to 3 and since z_{UB} is also equal to 3, we have closed the gap and we have proved the optimality of the current best feasible solution $x^* = (1, 0, 1, 0, 1)$.