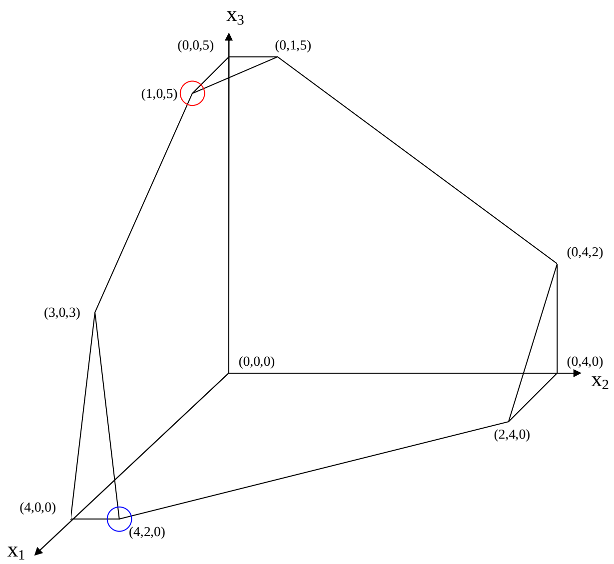
# 1 Exercise 1

Consider the following linear programming problem.

$$\begin{array}{rcl} \max z = 2x_1 + \ x_2 + 3x_3 & & \\ x_1 + \ x_2 + \ x_3 & \leq & 6 \\ x_2 \qquad & \leq & 4 \\ x_3 & \leq & 5 \\ 3x_1 \qquad + \ x_3 & \leq & 12 \\ x_1, x_2, x_3 & \geq & 0 \end{array}$$

The geometrical representation of the polyhedron is as follows.




If we put the problem in standard form and we write the initial tableau we obtain: Assume we do not have enough space to store this tableau with 4+1 rows and 7+1

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | -2 | -1 | -3 | 0 | 0 | 0 | 0 |
| 6 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 6 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 5 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| 12 | 3 | 0 | 1 | 0 | 0 | 0 | 1 |
| | N | N | N | B | B | B | B |

columns. Assume we have to neglect at least one column.

If we exclude column 3 from the tableau, we forbid $x_3$ to become basic and we solve a restricted LP problem, whose feasible region is the facet of the polyhedron $x_3 = 0$.

The restricted tableau is obtained from the previous tableau by deleting column 3. To represent this, we put all entries of column 3 between parentheses. Note that also

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | -2 | -1 | (-3) | 0 | 0 | 0 | 0 |
| 6 | 1 | 1 | ( 1) | 1 | 0 | 0 | 0 |
| 6 | 0 | 1 | ( 0) | 0 | 1 | 0 | 0 |
| 5 | 0 | 0 | ( 1) | 0 | 0 | 1 | 0 |
| 12 | 3 | 0 | ( 1) | 0 | 0 | 0 | 1 |
| | N | N | ( N) | B | B | B | B |

constraint 3 is now redundant and could be deleted from the restricted tableau.

For the first step we select column 1, because it is the one with the minimum reduced cost among those belonging to the restricted tableau. The pivot is on row 4 and it is indicated in red. After pivoting the following tableau is obtained. In the second

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 0 | -1 | $(-\frac{7}{3})$ | 0 | 0 | 0 | $\frac{2}{3}$ |
| 2 | 0 | 1 | $(\frac{2}{3})$ | 1 | 0 | 0 | $-\frac{1}{3}$ |
| 6 | 0 | 1 | ( 0) | 0 | 1 | 0 | 0 |
| 5 | 0 | 0 | ( 1) | 0 | 0 | 1 | 0 |
| 4 | 1 | 0 | $(\frac{1}{3})$ | 0 | 0 | 0 | $\frac{1}{3}$ |
| | B | N | ( N) | B | B | B | N |

step we select column 2, because it has negative reduced cost. The pivot is on row 1 and it is indicated in red. After pivoting the following tableau is obtained. Now the

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10 | 0 | 0 | $(-\frac{5}{3})$ | 1 | 0 | 0 | $\frac{1}{3}$ |
| 2 | 0 | 1 | $(\frac{2}{3})$ | 1 | 0 | 0 | $-\frac{1}{3}$ |
| 4 | 0 | 0 | $(-\frac{2}{3})$ | -1 | 1 | 0 | $\frac{1}{3}$ |
| 5 | 0 | 0 | ( 1) | 0 | 0 | 1 | 0 |
| 4 | 1 | 0 | $(\frac{1}{3})$ | 0 | 0 | 0 | $\frac{1}{3}$ |
| | B | B | ( N) | N | B | B | N |

optimality conditions in the restricted tableau are satisfied. Solution $(4, 2, 0)$ is indeed optimal (it has value 10) in the projected polyhedron. However we can improve if we allow $x_3$ entering the basis. We cannot realize this from the restricted tableau because column 3 is not there. But assume we have stored the initial column 3 in a suitable data-structure. The column has $c_3 = 3$; hence after rewriting the objective function in minimization form, it would have initial reduced cost $r_3 = -c_3 = -3$. At the generic iteration we have $r_3 = -c_3 + (a_{13}\lambda_4 + a_{23}\lambda_5 + a_{33}\lambda_6 + a_{43}\lambda_7)$. The coefficients on column 3 are $a_{13} = 1$, $a_{23} = 0$, $a_{33} = 1$, $a_{43} = 1$. We know from the restricted tableau the current values of the dual variables. They are $\lambda_4 = 1$, $\lambda_5 = 0$, $\lambda_6 = 0$ and

$\lambda_7 = \frac{1}{3}$. Hence we can compute the reduced cost of column 3:

$$r_3 = -c_3+(a_{13}\lambda_4+a_{23}\lambda_5+a_{33}\lambda_6+a_{43}\lambda_7) = -3+(1\times1+0\times0+1\times0+1\times\frac{1}{3}) = -3+\frac{4}{2} = -\frac{5}{3}.$$

Since the reduced cost is negative, we know it is convenient to insert column 3 into the tableau. We have to delete one column to save room; we decide to delete column 4, because its non-basic and its reduced cost is "high". Neglecting column 4 corresponds to forbid $x_4$ to become basic. In other terms we are forcing constraint $x_1+x_2+x_3 \leq 6$ to be active. Hence we are now intersecting the polyhedron with the plane $x_1 + x_2 + x_3 = 6$.

The pivot to be used on column 3 is on row 1. After pivoting the following tableau is obtained.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 15 | 0 | $\frac{5}{2}$ | 0 | $(\frac{7}{2})$ | 0 | 0 | $-\frac{1}{2}$ |
| 3 | 0 | $\frac{3}{2}$ | 1 | $(\frac{3}{2})$ | 0 | 0 | $-\frac{1}{2}$ |
| 6 | 0 | 1 | 0 | (0) | 1 | 0 | 0 |
| 2 | 0 | $-\frac{3}{2}$ | 0 | $(-\frac{3}{2})$ | 0 | 1 | $\frac{1}{2}$ |
| 3 | 1 | $-\frac{1}{2}$ | 0 | $(-\frac{1}{2})$ | 0 | 0 | $\frac{1}{2}$ |
| | B | N | B | (N) | B | B | N |

Optimality conditions are not satisfied. We need another pivot step. The pivot element is on column 7, row 3. The resulting tableau is the following.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | 0 | 1 | 0 | (2) | 0 | 1 | 0 |
| 5 | 0 | 0 | 1 | (0) | 0 | 1 | 0 |
| 6 | 0 | 1 | 0 | (0) | 1 | 0 | 0 |
| 4 | 0 | -3 | 0 | (-3) | 0 | 2 | 1 |
| 1 | 1 | 1 | 0 | (1) | 0 | -1 | 0 |
| | B | N | B | (N) | B | N | B |

Now we have reached the optimal solution with respect to the projected polyhedron. For being sure it is also optimal for the original problem we must check the reduced cost of the neglected column, i.e. column 4. The column has $c_4 = 0$; after rewriting the objective function in minimization form, it has initial reduced cost $r_4 = -c_4 = 0$. At the generic iteration we have $r_4 = -c_4 + (a_{14}\lambda_4 + a_{24}\lambda_5 + a_{34}\lambda_6 + a_{44}\lambda_7)$. The coefficients on column 4 are $a_{14} = 1$, $a_{24} = 0$, $a_{34} = 0$, $a_{44} = 0$. We know from the restricted tableau the current values of the dual variables. They are $\lambda_4 = 2$, $\lambda_5 = 0$, $\lambda_6 = 1$ and $\lambda_7 = 0$. Hence we can compute the reduced cost of column 4:

$$r_4 = -c_4+(a_{14}\lambda_4+a_{24}\lambda_5+a_{34}\lambda_6+a_{44}\lambda_7) = 0+(1\times2+0\times0+0\times1+0\times0) = 0+2 = 2.$$

Since the reduced cost is positive, we know the current solution is also optimal for the complete tableau, i.e. for the original problem.