Gomory cuts in branch-and-cut algorithms

Operational Research Complements

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Branch-and-cut

Sub-problems generated by branching in a branch-and-bound algorithm are more and more restricted: the polyhedron of the linear relaxation of each node is contained in the polyhedron of the linear relaxation of its precedessor node.

A cut generated at a node P is guaranteed to be valid also for all sub-problems in the sub-tree rooted at P. This is not true in general for the nodes that do not belong to the subtree rooted at P.

For many years this was considered a main obstacle hampering the use of Gomory cuts in branch-and-cut algorithms.

However, in the special case of mixed 0-1 linear programming it is possible to generate Gomory cuts that are valid for the whole tree.

Example

$$
\begin{aligned}
MILP) \text{ minimize } z =& 3x_1 + x_2 + 3x_3 + 4x_4 \\
\text{s.t. } & 2x_1 + 3x_2 + x_3 + x_4 = 4 \\
& x_1, x_2, x_3 \in \{0, 1\} \\
& x_4 \geq 0
\end{aligned}
$$

The optimal solution of the linear relaxation is

$$x^*_{LP} = \left[\frac{1}{2} \; 1 \; 0 \; 0\right]$$

$$z^*_{LP} = \frac{5}{2}$$

Then, branching occurs on x_1 which is fractional.

Example

$$(MILP)\ \text{minimize } z = 3x_1 + x_2 + 3x_3 + 4x_4$$
$$\text{s.t. } 2x_1 + 3x_2 + x_3 + x_4 = 4$$
$$x_1, x_2, x_3 \in \{0, 1\}$$
$$x_4 \geq 0$$

After fixing $x_1 = 1$ we have:

$$\text{minimize } z = 3 + x_2 + 3x_3 + 4x_4$$
$$\text{s.t. } 3x_2 + x_3 + x_4 = 2$$
$$x_2, x_3 \in \{0, 1\}$$
$$x_4 \geq 0$$

The optimal solution of the linear relaxation is

$$x^*_{LP} = \left[(1)\ \frac{2}{3}\ 0\ 0\right]$$

The Gomory cut generated from constraint

$$x_2 = \frac{2}{3} - \frac{1}{3}x_3 - \frac{1}{3}x_4$$

is

$$\frac{1}{3}x_3 + \frac{1}{3}x_4 \geq \frac{2}{3}$$

which is valid when $x_1 = 1$ but is not valid when $x_1 = 0$.

Notation

Consider a generic node in the B&B tree. We use the following notation:

- F_0: index set of the variables fixed at 0 by branching,
- F_1: index set of the variables fixed at 1 by branching,
- $\overline{a}_{ij}$: coefficient on row i, column j in the tableau of the optimal solution of the linear relaxation,
- B: index set of the basic variables,
- N: index set of the non-basic variables.

We also assume that the variables are numbered so that

- variables $x_1, \ldots, x_p$ are binary (in the relaxation they range in $[0, 1]$)
- variables with index larger than p are continuous and non-negative.

Notation

The constraints set of a generic (relaxed) sub-problem in standard form in the B&B tree is:

$$
\begin{aligned}
x_i &= \overline{a}_{i0} + \sum_{j \in N} \overline{a}_{ij}(-x_j) && \forall i \in B \\
x_k &\geq 0 && \forall k \in B \cup N \\
x_k &\leq 0 && \forall k \in F_0 \\
x_k &\geq 1 && \forall k \in F_1
\end{aligned}
$$

We can assume that all fixed variables have been fixed to 0.
Fixing a variable to 1 is equivalent to fixing its complement to 0.

The main result

Theorem. For any $i \in B$ with $i \leq p$ the cut $\gamma x \geq 1$ cuts off x^*_{LP} and is valid for *MILP*, where

$$\gamma_j = \begin{cases} \min\left\{\frac{f_{ij}}{f_{i0}}, \frac{1-f_{ij}}{1-f_{i0}}\right\} & \forall j \in N, j \leq p \\ \max\left\{\frac{\bar{a}_{ij}}{f_{i0}}, \frac{-\bar{a}_{ij}}{1-f_{i0}}\right\} & \forall j \in N, j \geq p+1 \\ 0 & \forall j \in B \end{cases}$$

Proof: notation

For proving the theorem we need the following notation to partition the non-basic variables into four subsets, once a row $i \in B$ has been selected such that x_i^* is fractional, i.e. $f_{i0} > 0$:

- $N_1 = N \cap \{1, \dots, p\}$
- $N_2 = N \setminus N_1$
- $N_1^+ = \{j \in N_1 : f_{ij} < f_{i0}\}$
- $N_1^- = N_1 \setminus N_1^+$
- $N_2^+ = \{j \in N_2 : \overline{a}_{ij} > 0\}$
- $N_2^- = N_2 \setminus N_2^+$

Proof: step 1

Assume $F_0 = \emptyset$, i.e. no variables fixed.

By definition of integral and fractional part of a number,

$$\overline{a}_{ij} = \lfloor \overline{a}_{ij} \rfloor + f_{ij} \qquad \forall j \in N_1^+ \tag{1}$$

$$-\overline{a}_{ij} = \lfloor -\overline{a}_{ij} \rfloor + 1 - f_{ij} \qquad \forall j \in N_1^-. \tag{2}$$

The constraint

$$x_i = \overline{a}_{i0} + \sum_{j \in N} \overline{a}_{ij}(-x_j) \tag{3}$$

can be rewritten as

$$f_{i0} = \left[\sum_{j \in N_1^+} f_{ij} x_j + \sum_{j \in N_1^-} (f_{ij} - 1) x_j + \sum_{j \in N_2^+} \overline{a}_{ij} x_j + \sum_{j \in N_2^-} \overline{a}_{ij} x_j \right] \mod 1. \tag{4}$$

Proof: step 2

From

$$f_{i0} = \left[\sum_{j \in N_1^+} f_{ij} x_j + \sum_{j \in N_1^-} (f_{ij} - 1) x_j + \sum_{j \in N_2^+} \overline{a}_{ij} x_j + \sum_{j \in N_2^-} \overline{a}_{ij} x_j \right] \mod 1.$$

it follows that at least one of these two inequalities must be satisfied:

$$\sum_{j \in N_1^+} f_{ij} x_j + \sum_{j \in N_2^+} \overline{a}_{ij} x_j \geq f_{i0} \tag{5}$$

$$\sum_{j \in N_1^-} (f_{ij} - 1) x_j + \sum_{j \in N_2^-} \overline{a}_{ij} x_j \leq f_{i0} - 1 \tag{6}$$

because

- the left-hand-side coefficients in (5) are all non-negative,
- the left-hand-side coefficients in (6) are all non-positive,
- all variables are non-negative.

Proof: step 3

After reversing the second inequality and dividing both inequalites by their right-hand-side, one obtains

$$\sum_{j \in N_1^+} \frac{f_{ij}}{f_{i0}} x_j + \sum_{j \in N_2^+} \frac{\overline{a}_{ij}}{f_{i0}} x_j \geq 1 \tag{7}$$

$$\sum_{j \in N_1^-} \frac{1 - f_{ij}}{1 - f_{i0}} x_j + \sum_{j \in N_2^-} \frac{-\overline{a}_{ij}}{1 - f_{i0}} x_j \geq 1. \tag{8}$$

All left-hand side coefficients are non-negative: hence, both left-hand-sides are non-negative.
Since at least one of the inequalities is satisfied, the sum of the two left-hand-sides is guaranteed to be ≥ 1.
Therefore:

$$\sum_{j \in N_1^+} \frac{f_{ij}}{f_{i0}} x_j + \sum_{j \in N_1^-} \frac{1 - f_{ij}}{1 - f_{i0}} x_j + \sum_{j \in N_2^+} \frac{\overline{a}_{ij}}{f_{i0}} x_j + \sum_{j \in N_2^-} \frac{-\overline{a}_{ij}}{1 - f_{i0}} x_j \geq 1. \tag{9}$$

Proof: step 3

- For $j \in N_1^+$, since $f_{ij} < f_{i0}$, then $\frac{f_{ij}}{f_{i0}} < \frac{1 - f_{ij}}{1 - f_{i0}}$.
- For $j \in N_1^-$, since $f_{ij} \geq f_{i0}$, then $\frac{f_{ij}}{f_{i0}} \geq \frac{1 - f_{ij}}{1 - f_{i0}}$.
- Hence forall $j \in N_1$, $\gamma_j = \min\left\{\frac{f_{ij}}{f_{i0}}, \frac{1 - f_{ij}}{1 - f_{i0}}\right\}$.
- For $j \in N_2^+$, since $\overline{a}_{ij} > 0$, then $\frac{\overline{a}_{ij}}{f_{i0}} > \frac{-\overline{a}_{ij}}{1 - f_{i0}}$.
- For $j \in N_2^-$, since $\overline{a}_{ij} \leq 0$, then $\frac{\overline{a}_{ij}}{f_{i0}} \leq \frac{-\overline{a}_{ij}}{1 - f_{i0}}$.
- Hence forall $j \in N_2$, $\gamma_j = \max\left\{\frac{\overline{a}_{ij}}{f_{i0}}, \frac{-\overline{a}_{ij}}{1 - f_{i0}}\right\}$.

Therefore cut (9) is $\gamma x \geq 1$ as defined in the theorem statement.
This concludes the proof for the case $F_0 = \emptyset$.

Proof: $F_0 \neq \emptyset$

Steps 1 and 3 involve only algebraic manipulations and their validity is not affected by the possible presence of variables fixed at 0.

The theorem is valid also when $F_0 \neq \emptyset$, because Step 2 requires only that all variables are non-negative in the linear relaxation.
If some variables have been fixed to 0 by constraints with the form $x_k \leq 0$, the cut $\gamma x \geq 1$ remains valid also without the variable fixing constraint, because x_k is non-negative also in the other sub-problems of the B&B tree .

Unfortunately, this argument no longer holds in case of MILPs, because the slack variables of a variable fixing constraint, like $x_k \leq \lfloor x_k^* \rfloor$ or $x_k \geq \lceil x_k^* \rceil$, are not guaranteed to be non-negative in other sub-problems, where the variable x_k is not bounded.

Cuts generation strategies

In general, it seems profitable to generate a Gomory cut from each fractional basic variable before reoptimizing the linear relaxation, instead of a single cut.

However, it may pay off not to generate Gomory cuts after a single branching, but once every σ nodes are enumerated, where σ is called *skip factor*. A heuristic rule of thumb to set it is

$$\sigma = \min\{\overline{\sigma}, \left\lceil \frac{f}{c\, d\, \log_{10} p} \right\rceil\},$$

where

- f is the number of fractional variables in x^*_{LP} at the root node,
- d is the average Euclidean distance between the generated cuts and x^*_{LP} at the root node,
- c and $\overline{\sigma}$ are two parameters.

Cuts management

All generated cuts are kept in a unique pool. For each sub-problem the pool is scanned to search for cuts that are either tight or violated at the optimal solution of the predecessor node.

When Gomory cuts are generated, the current LP is also “cleaned” by deleting all Gomory cuts generated in previous iterations that are not active at the current optimal solution. This keeps the size of the tableau under control and helps avoiding numerical instabilities.