Lifting

Operational Research Complements

Giovanni Righini

Università degli Studi di Milano

Based on: H. Marchand, A. Martin, R. Weismantel, L. Wolsey, *Cutting planes in integer and mixed-integer programming*, Discrete Applied Mathematics 123 (2002) 397-446.

Extended valid inequalities

Valid inequalities can be made stronger by *extension* and *lifting*.

Valid inequalities are extended by inserting additional variables in their left-hand-side.

Example (Knapsack problem):

$$\begin{aligned} \text{maximize } z = & \sum_{j \in N} c_j x_j \\ \text{s.t. } & \sum_{j \in N} a_j x_j \le b \\ & x \in \mathcal{B}^{|N|} \end{aligned}$$

All *cover inequalities* are valid:

$$\sum_{j \in C} x_j \le |C| - 1 \quad \forall C \subseteq N : \sum_{j \in C} a_j > b.$$

Example

For instance:

$$K = \{x \in \mathcal{B}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$$

Some minimal covers are:

$$\begin{array}{lllr}
\{1,2,3\} & x_1 + x_2 + x_3 & & \leq 2 \\
\{1,2,6\} & x_1 + x_2 & x_6 & \leq 2 \\
\{1,5,6\} & x_1 & x_5 + x_6 & \leq 2 \\
\{3,4,5,6\} & & x_3 + x_4 + x_5 + x_6 & \leq 3
\end{array}$$

Example

For each cover C we have an extended cover inequality:

$$\sum_{j \in E(C)} x_j \leq |C| - 1 \quad \forall C \subseteq N : \sum_{j \in C} a_j > b$$

where

$$E(C) = C \cup \{j \in N : a_j \geq a_i \ \forall i \in C\}.$$

For instance:

$$C = \{3, 4, 5, 6\} \to x_3 + x_4 + x_5 + x_6 \leq 3.$$

$$E(C) = \{1, 2, 3, 4, 5, 6\} \to x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3.$$

We insert two more variables x_1 and x_2 because their weights (11 and 6) are not smaller than those of the variables in C $(6, 5, 5, 4)$.

Lifting: example

Lifting consists of inserting variables in the left-hand-side of a valid inequality, giving them the best possible coefficient.

Example (cover inequality):

$$C = \{3, 4, 5, 6\} \rightarrow \pi_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3.$$

What is the maximum value of π_1 for which this inequality still valid? The inequality

$$\pi_1 \; 1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

must be valid for all integer points such that

$$11 \; 1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19.$$

To find the maximum value of π_1, we must solve an auxiliary (smaller) discrete optimization problem:

$$\begin{aligned} \text{maximize } \sigma = & x_3 + x_4 + x_5 + x_6 \\ \text{s.t. } & 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8 \\ & x \in \mathcal{B} \end{aligned}$$

Lifting: example

We find $\sigma = 1$ and therefore $\pi_1 \leq 3 - \sigma = 2$.
The lifted cover inequality is

$$2x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

The procedure can now be repeated to lift another variable into the inequality.

$$2x_1 + \pi_2 x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$$

It must be $\pi_2 \leq 3 - \sigma$, where

$$\begin{aligned} \text{maximize } \sigma = & 2x_1 + x_3 + x_4 + x_5 + x_6 \\ \text{s.t. } & 11x_1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 13 \\ & x \in \mathcal{B} \end{aligned}$$

We find $\sigma = 2$ and hence we can set $\pi_2 = 1$ and obtain

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$$

(that dominates the extended cover inequalities obtained from the same cover).

Lifting: formal definition

Consider a 0-1 Knapsack polyhedron

$$K = \{x \in \{0,1\}^{|N|} : \sum_{j \in N} a_j x_j \leq b\}$$

and a subset $M \subseteq N$. Assume we know a valid inequality

$$\sum_{j \in M} \pi_j x_j \leq \pi_0$$

valid for $K_M = K \cap \{x : x_j = 0 \; \forall j \in N \backslash M\}$.

We want to compute coefficients $\pi_j \; \forall j \in N \backslash M$ such that

$$\sum_{j \in N} \pi_j x_j \leq \pi_0$$

be valid for K and we want such coefficients to be as large as possible, in order to make the resulting inequality as strong as possibile.

Sequential lifting

We select a variable to be lifted: $x_k : k \in N \backslash M$. We want

$$\pi_k x_k + \sum_{j \in M} \pi_j x_j \leq \pi_0$$

be valid for $K_{M \cup \{k\}}$.

We define the following lifting function:

$$\Phi_M(u) = \min\{\pi_0 - \sum_{j \in M} \pi_j x_j : \sum_{j \in M} a_j x_j \leq b - u, x \in \{0,1\}^{|M|}\}.$$

Assume $K_{M \cup \{k\}} \neq \emptyset$.

Then $\pi_k x_k + \sum_{j \in M} \pi_j x_j \leq \pi_0$ is valid for $K_{M \cup \{k\}}$ if

$$\pi_k \leq \Phi_M(a_k).$$

Furthermore, if we select $\pi_k = \Phi_M(a_k)$ and $\sum_{j \in M} \pi_j x_j \leq \pi_0$ defines a face of dimension t of $conv(K_M)$, then the lifted inequality defines a face of dimension at least $t + 1$ of $conv(K_{M \cup \{k\}})$.

Sequential lifting

After lifting x_k into the inequality, the lifting function

$$\Phi_M(u) = \min\{\pi_0 - \sum_{j \in M} \pi_j x_j : \sum_{j \in M} a_j x_j \leq b - u, x \in \{0,1\}^{|M|}\}$$

must be replaced by

$$\Phi_{M \cup \{k\}}(u) = \min\{\pi_0 - \sum_{j \in M \cup \{k\}} \pi_j x_j : \sum_{j \in M \cup \{k\}} a_j x_j \leq b - u, x \in \{0,1\}^{|M|+1}\}.$$

Therefore the function decreases every time a variable is lifted.

The values of the coefficients obtained in this way depend on the order in which the variables are lifted.

The earlier a variable is lifted, the larger its coefficient can be.

From the same valid inequality, many lifted inequalities are obtained using different sequences.

Sequential lifting: example

$$K = \{x \in \{0,1\}^6 : 5x_1 + 5x_2 + 5x_3 + 5x_4 + 3x_5 + 8x_6 \leq 17\}$$

A cover is $M = \{1,2,3,4\}$. A cover inequality for K_M is

$$x_1 + x_2 + x_3 + x_4 \leq 3.$$

Lifting x_5:

$$\begin{aligned} \min\ z =& 3 - (x_1 + x_2 + x_3 + x_4) \\ s.t.& 5x_1 + 5x_2 + 5x_3 + 5x_4 \leq 14 \\ & x \text{ binary} \end{aligned}$$

which yields $\pi_5 = 1$.
Lifting x_6:

$$\begin{aligned} \min\ z =& 3 - (x_1 + x_2 + x_3 + x_4 + x_5) \\ s.t.& 5x_1 + 5x_2 + 5x_3 + 5x_4 + 3x_5 \leq 9 \\ & x \text{ binary} \end{aligned}$$

which yields $\pi_6 = 1$.

Sequential lifting: example

Following a different order, different coefficients are obtained.

Lifting x_6:

$$\begin{aligned} \min\ z =& 3 - (x_1 + x_2 + x_3 + x_4) \\ & s.t. 5x_1 + 5x_2 + 5x_3 + 5x_4 \leq 9 \\ & x \text{ binary} \end{aligned}$$

which yields $\pi_6 = 2$.
Lifting x_5:

$$\begin{aligned} \min\ z =& 3 - (x_1 + x_2 + x_3 + x_4 + 2x_6) \\ & s.t. 5x_1 + 5x_2 + 5x_3 + 5x_4 + 8x_6 \leq 14 \\ & x \text{ binary} \end{aligned}$$

which yields $\pi_5 = 0$.

Simultaneous lifting

To simultaneously lift all coefficients of the missing variables into a valid inequality, we need a lifting function $\Psi : \Re \mapsto \Re$ such that

$$\sum_{j \in N \setminus M} \Psi(a_j)x_j + \sum_{j \in M} \pi_j x_j \leq \pi_0$$

is valid for K.

Definition. A function $F : \Re \mapsto \Re$ is superadditive on $\Re$ if

$$F(d_1) + F(d_2) \leq F(d_1 + d_2) \; \forall d_1, d_2 \in \Re.$$

Sufficient condition (Wolsey, 1977; Gu, Nemhauser, Salvesbergh, 2000) for $\sum_{j \in N \setminus M} \Psi(a_j)x_j + \sum_{j \in M} \pi_j x_j \leq \pi_0$ to be valid for K is

1. $\Psi(u) \leq \Phi_M(u) \; \forall u \in \Re$.
2. $\Psi(u)$ is superadditive on $\Re$.

If $a_k \geq 0 \; \forall k \in N$ then it is enough that $\Psi(u)$ is superadditive on $\Re_+$.

Lifted cover inequalities with a superadditive function

Given a Knapsack polyhedron K with $a_j > 0 \; \forall j \in N$, and given a minimal cover $C \subseteq N$,

$$\sum_{j \in C} x_j \leq |C| - 1$$

is valid for $K_C = K \cap \{x : x_j = 0 \; \forall j \in N \backslash C\}$.

The sequential lifting function is

$$\Phi_C(u) = \min\{|C| - 1 - \sum_{j \in C} x_j : \sum_{j \in C} a_j x_j \leq b - u, x \in \{0,1\}^{|C|}\}.$$

Assume $C = \{1, \ldots, r\}$ and $a_j \geq a_{j+1} \; \forall j = 1, \ldots, r-1$.
Let $A_j = \sum_{t=1}^{j} a_t \; \forall j = 1, \ldots, r$, $A_0 = 0$, $\lambda = A_r - b$.
A superadditive function for simultaneous lifting is

$$\Psi(u) = \begin{cases} j & \text{if } A_j \leq u \leq A_{j+1} - \lambda \; \forall j = 0, \ldots, r-1 \\ j + \frac{u - A_j}{\lambda} & \text{if } A_j - \lambda \leq u \leq A_j \; \forall j = 1, \ldots, r-1 \\ r + \frac{u - A_r}{\lambda} & \text{if } A_r - \lambda \leq u \end{cases}$$

Example

$$K = \{x \in \{0,1\}^6 : 5x_1 + 5x_2 + 5x_3 + 5x_4 + 3x_5 + 8x_6 \leq 17\}.$$

Cover $\{1,2,3,4\}$ yields the cover inequality

$$x_1 + x_2 + x_3 + x_4 \leq 3.$$

with $\lambda = 3$ and $A = [0\ 5\ 10\ 15\ 20]$ for $j = 0, \ldots, 4$.

By simultaeous lifting one obtains

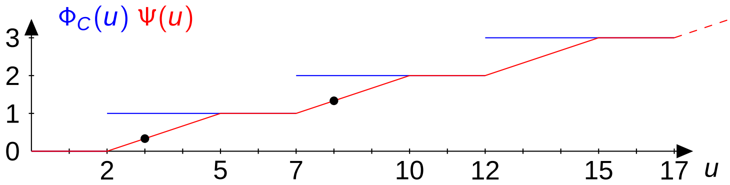
$$x_1 + x_2 + x_3 + x_4 + \frac{1}{3}x_5 + \frac{4}{3}x_6 \leq 3.$$

The lifting function for sequential lifting is

$$\Phi_C(u) = \min\{3 - (x_1 + x_2 + x_3 + x_4) : 5x_1 + 5x_2 + 5x_3 + 5x_4 \le 17 - u$$
$$x \in \{0,1\}^4\}$$

The superadditive function for simultaneous lifting is

$$\Psi(u) = \begin{cases} 0 & \text{if } 0 \le u \le 5-3 \\ 1 + \frac{u-5}{3} & \text{if } 5-3 \le u \le 5 \\ 1 & \text{if } 5 \le u \le 10-3 \\ 2 + \frac{u-10}{3} & \text{if } 10-3 \le u \le 10 \\ 2 & \text{if } 10 \le u \le 15-3 \\ 3 + \frac{u-15}{3} & \text{if } 15-3 \le u \le 15 \\ 3 & \text{if } 15 \le u \le 20-3 \\ 4 + \frac{u-20}{3} & \text{if } 20-3 \le u \end{cases}$$



Feasible set inequalities

Consider a knapsack set K, as before, with $a_j > 0\ \forall j \in N$.
Assign an arbitrary weight $w_j > 0$ with each variable $j \in N$.

Definition. A subset $T \subset N$ is a feasible set if and only if

$$\sum_{j \in T} a_j \le b.$$

We indicate the non-negative slack as $r_T = b - \sum_{j \in T} a_j$.
We also define $w(T) = \sum_{j \in T} w_j$.
The inequality

$$\sum_{j \in T} w_j x_j \le w(T)$$

is valid for $K_T = K \cap \{x : x_j = 0\ \forall j \in N \backslash T\}$.

Feasible set inequalities

Consider a permutation $\mu_1, \mu_2, \ldots, \mu_{|N|-|T|}$ of the variables not in T.

We carry out sequential lifting with the lifting function

$$\Phi_T(u) = \min\{w(T) - \sum_{j \in T} w_j x_j : \sum_{j \in T} a_j x_j \leq b - u, x \in \{0,1\}^{|T|}\}$$

and we obtain

$$\pi_{\mu_i} = \Phi_{T \cup \{\mu_1, \ldots, \mu_{i-1}\}}(a_{\mu_i}).$$

Then the feasible set inequality

$$\sum_{j \in T} w_j x_j + \sum_{j \in N \setminus T} \pi_j x_j \leq w(T)$$

is valid for K (Weismantel, 1997).

Feasible set inequalities

A special case occurs when we select $w_j = 1 \; \forall j \in N$ and hence $w(T) = |T|$. In this case the lifting function can be restated as follows:

$$\Phi_T(u) = \min\{|S| : S \subseteq T, \sum_{j \in S} a_j \geq u - r_T\}.$$

Intuitively, it represents the minimum number of elements that must be deleted from T to satisfy the capacity constraint when an additional element of weight u is inserted.

In this special case the following property holds

$$\Phi_T(a_j) - 1 \leq \pi_j \leq \Phi_T(a_j)$$

and this makes the computation of π_j values more efficient.

Example

Consider the knapsack constraint

$$3x_1 + 4x_2 + 6x_3 + 7x_4 + 9x_5 + 18x_6 \leq 21$$

and the feasible subset $T = \{1, 2, 3, 4\}$.
We have $r_T = 1$ and $w(T) = |T| = 4$.
Consider the permutation $\mu = (5, 6)$.

Lifting 5, we obtain

$$\pi_5 = \Phi_{\{1,\ldots,4\}}(9) = \min\{|S| : S \subseteq \{1, \ldots, 4\}, \sum_{j \in S} a_j \geq 9 - 1\} = 2.$$

Lifting 6, we obtain

$$\pi_6 = \Phi_{\{1,\ldots,5\}}(18) = \min\{|S| : S \subseteq \{1, \ldots, 5\}, \sum_{j \in S} a_j \geq 18 - 1\} = 3.$$

The resulting feasible set inequality is

$$x_1 + x_2 + x_3 + x_4 + 2x_5 + 3x_6 \leq 4$$

$(1, k)$-configuration inequalities

Feasible set inequalities with unit weights $w_j = 1 \; \forall j \in N$ generalize lifted cover inequalities and $(1, k)$-configuration inequalities.

Definition. A pair (T, z) with $T \cup \{z\} \subseteq N$ and $\sum_{j \in T} a_j \leq b$ is a $(1, k)$-configuration if $S \cup \{z\}$ is a minimal cover for all $S \subseteq T : |S| = k$.

Then

$$\sum_{j \in T} x_j + (|T| - k + 1) x_z \leq |T|$$

is valid for K.