Special cutting planes for ILP and MILP

Operational Research Complements

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Cover inequalities (0-1 programming)

Consider the 0-1 Knapsack set:

$$X = \left\{ x \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \leq b \right\}$$

with $a_j \geq 0 \ \forall j = 1, \ldots, n$ and $b \geq 0$.

Definition. The subset $C \subseteq \{1, \ldots, n\}$ is a cover if and only if

$$\sum_{j \in C} a_j > b.$$

C is minimal if and only if it does not contain any other cover, i.e.

$$a_k \geq \sum_{j \in C} a_j - b \quad \forall k \in C.$$

Cover inequalities

With each cover C we can associate a valid cover inequality:

$$\sum_{j \in C} x_j \leq |C| - 1$$

meaning that not all binary variables in the cover may be equal to 1 in any feasible solution.

If C is minimal, than the corresponding cover inequality is also facet-defining for $X_C = X \cap \{x \in \{0,1\}^n : x_j = 0 \; \forall j \notin C\}$.

If C is not minimal, then the corresponding cover inequality is redundant.

Example

Consider

$$X = \{x \in \{0,1\}^6 : 5x_1 + 5x_2 + 5x_3 + 5x_4 + 3x_5 + 8x_6 \leq 17\}.$$

The subset $C = \{1,2,3,4\}$ is a minimal cover.

The corresponding cover inequality

$$x_1 + x_2 + x_3 + x_4 \leq 3$$

is facet-defining for

$$X_C = \{x \in \{0,1\}^4 : 5x_1 + 5x_2 + 5x_3 + 5x_4 \leq 17\}$$

Cover inequalities

Facets of X can be obtained from facets of X_C by a technique called lifting.

Cover inequalities are commonly used by MILP solvers to strengthen formulations.

Balas and Zemel (1984) proved that minimal covers, lifting and complementation (replacing a binary variable x_j by its complement $1 - x_j$) can be used to obtain all the non-trivial facets of any 0-1 programming polytope with positive coefficients.

Cover inequalities (mixed 0-1 programming)

Consider the mixed 0-1 set X with a single continuous variable:

$$X = \left\{ (x, s) \in \{0, 1\}^n \times \Re_+ : \sum_{j=1}^{n} a_j x_j \leq b + s \right\}$$

with $a_j \geq 0 \ \forall j = 1, \ldots, n$ and $b \geq 0$.

Consider a cover $C \subseteq \{1, \ldots, n\}$ and define $\lambda_C = \sum_{j \in C} a_j - b$.

Marchand and Wolsey (1999) proved that

$$\sum_{j \in C} \min\{a_j, \lambda_C\} x_j \leq \sum_{j \in C} \min\{a_j, \lambda_C\} - \lambda_C + s \qquad (1)$$

is valid for X and it defines a facet of
$conv(X_C) = X \cap \{x \in \{0, 1\}^n : x_j = 0 \ \forall j \notin C\}$.

In this context the result holds for all (not only minimal) covers.

Example

Consider

$$X = \{x \in \{0,1\}^6 : 5x_1 + 5x_2 + 5x_3 + 5x_4 + 3x_5 + 8x_6 \leq 17 + s\}.$$

The subset $C = \{1,2,3,6\}$ is a (non-minimal) cover with $\lambda_C = 6$.

The corresponding mixed-integer cover inequality

$$5x_1 + 5x_2 + 5x_3 + 6x_6 \leq 15 + s$$

is valid for X and it is facet-defining for

$$X_C = \{x \in \{0,1\}^4 : 5x_1 + 5x_2 + 5x_3 + 8x_6 \leq 17 + s\}.$$

Flow cover inequalities

Consider the flow set

$$X = \{(x, y) \in \{0, 1\}^n \times \Re^n_+ : \sum_{j=1}^{n} y_j \leq b, y_j \leq a_j x_j \; \forall j = 1, \ldots, n\},$$

where $N = \{1, \ldots, n\}$ is a set of arcs, each one carrying flow to a node in a flow network, y_j is the amount of flow on each arc $j \in N$, b is an upper limit to the total inflow the node can receive, a_j is the capacity of arc $j \in N$ and x_j indicates whether arc $j \in N$ carries non-zero flow or not.

Let C be a flow cover, i.e. a subset of N such that $\sum_{j \in C} a_j > b$.

Flow cover inequalities

In the constraint

$$\sum_{j=1}^{n} y_j \leq b$$

let ignore the contribution of arcs not in the cover. We obtain

$$\sum_{j \in C} y_j \leq b. \tag{2}$$

Rewrite constraints

$$y_j \leq a_j x_j \ \forall j \in C$$

as

$$y_j + s_j = a_j x_j \ \forall j \in C$$

where $s_j \geq 0$ is a slack variable.

Flow cover inequalities

Now replacing y_j in (2) one obtains

$$\sum_{j \in C} a_j x_j \le b + \sum_{j \in C} s_j. \tag{3}$$

Using (1) and treating $\sum_{j \in C} s_j$ as a single aggregate continuous variable, we know that the inequality

$$\sum_{j \in C} \min\{a_j, \lambda_C\} x_j \le \sum_{j \in C} \min\{a_j, \lambda_C\} - \lambda_C + \sum_{j \in C} s_j$$

is valid. Replacing $s_j = a_j x_j - y_j$ one obtains

$$\sum_{j \in C} \min\{a_j, \lambda_C\} x_j \le \sum_{j \in C} \min\{a_j, \lambda_C\} - \lambda_C + \sum_{j \in C} a_j x_j - \sum_{j \in C} y_j$$

$$\sum_{j \in C} \left[\min\{a_j, \lambda_C\} - a_j\right] x_j + \sum_{j \in C} y_j \le \sum_{j \in C} \min\{a_j, \lambda_C\} - \lambda_C$$

Flow cover inequalities

From

$$\sum_{j\in C}\left[\min\{a_j,\lambda_C\}-a_j\right]\ x_j+\sum_{j\in C}y_j\le\sum_{j\in C}\min\{a_j,\lambda_C\}-\lambda_C$$

replacing $\lambda_C=\sum_{j\in C}a_j-b$, one obtains

$$\sum_{j\in C}\left[\min\{a_j,\lambda_C\}-a_j\right]\ x_j+\sum_{j\in C}y_j\le\sum_{j\in C}\min\{a_j,\lambda_C\}-\sum_{j\in C}a_j+b$$

$$\sum_{j\in C}\left[\min\{a_j,\lambda_C\}-a_j\right]\ x_j+\sum_{j\in C}y_j\le\sum_{j\in C}\left[\min\{a_j,\lambda_C\}-a_j\right]+b$$

$$\sum_{j\in C}\left[a_j-\min\{a_j,\lambda_C\}\right]\ (1-x_j)+\sum_{j\in C}y_j\le b$$

and finally the flow cover inequality is obtained:

$$\sum_{j\in C}\left[y_j+\max\{a_j-\lambda_C,0\}(1-x_j)\right]\le b.$$

Flow cover inequalities

Padberg, van Roy and Wolsey (1985) proved that for any cover C such that $\max_{j \in C}\{a_j\} > \lambda_C$ the flow cover inequality

$$\sum_{j \in C} \left[y_j + \max\{a_j - \lambda_C, 0\}(1 - x_j) \right] \leq b$$

is facet-defining for X.

Van Roy and Wolsey (1987) and Gu, Nemhauser, Savelsbergh (1999,2000) successfully used lifted flow cover inequalities in branch-and-cut algorithms for general mixed 0-1 programming.

Céria, Cordier, Marchand and Wolsey (1998) derived cover inequalities for pure integer problems. Unfortunately, in this case cover inequalities are not facet-defining in general.

Binary matrices

Integer and mixed integer programs often contain constraints with only 0-1 coefficients.

In addition, many MILP solvers automatically generate constraints with this characteristic in a pre-processing step.

Therefore it is of interest to study the polytopes described by these constraints, i.e. integer programs with 0-1 matrices, with the aim of deriving valid inequalities.

We consider

- set packing
- independence system
- set covering

The Set Packing problem

Let $A \in \{0,1\}^{m \times n}$ be a 0-1 matrix and $c \in \Re^n$.

The 0-1 program

$$\max\{c^T x : Ax \leq 1, x \in \{0,1\}^n\}$$

is called Set Packing Problem (SPP).

Every column $j = 1, \ldots, n$ of A can be viewed as the incidence vector of a subset F_j of the ground set $\{1, \ldots, m\}$.

The SPP requires to find a maximum weight collection of disjoint subsets.

The Max Independent Set problem

Associate a vertex of a graph $G(A) = (V(A), E(A))$ with each column of A and introduce an edge $[i, j]$ for each pair of vertices that share at least one element of the ground set i.e. $\exists k = 1, \ldots, m : A_{ki} = A_{kj} = 1$.

The resulting graph $G(A)$, with vertex set $V(A)$ and edge set $E(A)$, is called intersection graph.

Feasible solutions of the set packing problem are in 1-to-1 correspondence with independent sets of $G(A)$.

Hence, the Set Packing Problem is equivalent to the Max Independent Set Problem.

The Max Independent Set polytope

The max independent set polytope ($P(G(A))$) has the following properties:

- it is full-dimensional
- all non-trivial facets have non-negative coefficients
- the bounds $x_j \geq 0$ are facet-defining.

Edge constraints

$$x_i + x_j \leq 1 \ \ \forall [i, j] \in E(A)$$

and non-negativity constraints completely describe the polytope if and only if $G(A)$ is bipartite.

Odd cycle inequalities

Non-bipartite graphs contain odd cycles, that originate valid inequalites.

Let C be a subset of columns of A corresponding to an odd cycle in $G(A)$.

Padberg (1973) proved that the odd cycle inequality

$$\sum_{j \in V(C)} x_j \leq \frac{|V(C)| - 1}{2}$$

is valid. It defines a facet of $P(V(C), E(C))$ if and only if C is an odd hole, i.e. an odd cycle without chords.

Odd cycle inequalities can be separated in polynomial time (Grötschel, Lovász, Schrijver, 1988).

Clique inequalities

Let $(C, E(C))$ be a clique in the intersection graph G.

Fulkerson (1971) and Padberg (1973) proved that the clique inequality

$$\sum_{j \in C} x_j \leq 1$$

is valid for $P(G)$ and it is facet-defining if and only if C is maximal.

Intersection graphs for which the correspondinng polytope is completely described by clique inequalities are called perfect graphs.

The separation of clique inequalities is NP-hard.

Other set packing inequalities

There exists a larger class of inequalities, called orthonormal representation inequalities, that includes clique inequalities as a special case and that can be separated in polynomial time.

Other inequalities for the independent set polytope are known: blossom, odd antihole, wheel, antiweb and web, wedge inequalities and many more (Borndörfer, 1998).

Independence systems

An independence system generalizes, among others, the feasible sets of the 0-1 knapsack problem and the set packing problems.

Let N be a ground set and let $\mathcal{I}$ be a collection of subsets of N. Then $\mathcal{I}$ is an independence system if

$$F \in \mathcal{I} \;\Rightarrow\; G \in \mathcal{I} \quad \forall G \subseteq F.$$

Associated with an independence system $\mathcal{I}$ there is another collection $\mathcal{C}$ of subsets of N, called circuits.

It includes all subsets of N of minimal cardinality that do not belong to $\mathcal{I}$.

Knapsack problem: minimal covers are the circuits.
Stable set problem: edges are the circuits.

Circuit constraints

More generally, let $A \in \Re_{+}^{m \times n}$ a non-negative matrix and let $b \in \Re^{m}$ be a vector.

The set of all 0-1 solutions to $Ax \leq b$ form an independence system with a corresponding independence system polyhedron

$$P_{\mathcal{I}} = conv(\{x \in \{0, 1\}^{n} : Ax \leq b\}).$$

Property. Apart from lower bounds $x \geq 0$, all inequalities $\alpha^{T} x \leq \beta$ that are facet-defining of $P_{\mathcal{I}}$ have $\alpha \geq 0$ and $\beta > 0$.

Given any circuit C, the circuit constraint

$$\sum_{j \in C} x_{j} \leq |C| - 1$$

is valid for $P_{\mathcal{I}}$ (but not facet-defining, in general).

Rank inequalities

For any given subset $T \subseteq N$, the valid inequality

$$\sum_{j \in T} x_j \leq \max\{|S| : S \subseteq T, S \in \mathcal{I}\}$$

is called rank inequality.

In general it is *NP*-hard to compute the rank (the right-hand-side of the inequality).

Antiweb inequalities

Consider a set $\{1, \ldots, n\}$ and the set of adjacent t-tuples N_i of consecutive vertices, i.e. $\{i, i+1, \ldots, i+t-1\}$, (where indices are computed modulo n).

For instance (with $t = 4$): $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$, $\{3, 4, 5, 6\}$ and so on.

Assume it is allowed to select at most $q-1$ elements from each tuple, with $q \leq t$.

The resulting antiweb $\mathcal{AW}(n, t, q)$ is an independence system.

$$\mathcal{AW}(n, t, q) = \{I \subseteq N : |I \cap N_i| \leq q - 1 \ \forall i = 1, \ldots, n\}$$

The set of all circuits is

$$\mathcal{C} = \{C \subseteq N : |C| = q, \exists i \in \{1, \ldots, n\} : C \subseteq N_i\}.$$

Antiweb inequalities

Let $\mathcal{AW}(n, t, q)$ an antiweb and let $P_{\mathcal{I}}$ be the associated polyhedron. Then the antiweb inequality

$$\sum_{j \in N} x_j \leq \lfloor n(q-1)/t \rfloor$$

is valid for $P_{\mathcal{I}}$ (Laurent, 1989).

Proof. Summing up all constraints $\sum_{j \in N_i} x_j \leq q - 1$, we obtain the aggregate constraint

$$\sum_{j \in N} t\, x_j \leq n(q-1)$$

and, since x variables are binary, the right-hand-side can be rounded down.

An antiweb inequality is facet-defining if and only if $n(q-1)/t \notin \mathcal{Z}$.

No polynomial-time algorithms are known to separate antiweb inequalities.

Example

The antiweb $\mathcal{AW}(5,3,3)$ is

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

and the corresponding antiweb inequality is

$$\sum_{j \in N} x_j \leq 3.$$

The Set Covering problem

Let $A \in \{0,1\}^{m \times n}$ be a 0-1 matrix and $c \in \Re^n$. The 0-1 program

$$\min\{c^T x : Ax \geq 1, x \in \{0,1\}^n\}$$

is called Set Covering Problem (SCP).

It can be restated using complemented variables $\overline{x}_j = 1 - x_j$. The covering constraints are

$$\sum_{j \in N_i} \overline{x}_j \leq |N_i| - 1 \quad \forall i = 1, \ldots, n$$

meaning that it is forbidden to discard all columns covering row i.

These inequalities are analogous to cover inequalities for the knapsack problem.

Antiweb inequalities also have a counterpart for the Set Covering polytope.