General cutting planes for ILP and MILP

Operational Research Complements

Giovanni Righini

Università degli Studi di Milano

Based on: H. Marchand, A. Martin, R. Weismantel, L. Wolsey, *Cutting planes in integer and mixed-integer programming*, Discrete Applied Mathematics 123 (2002) 397-446.

Chvátal-Gomory cuts

A generic pure ILP is

$$\min\{c^T x : x \in X\} \quad X = \{z \in \mathcal{Z}_+^n : Ax = b\}$$

with A and b integer.

The linear relaxation is characterized by the polyhedron

$$P = \{x \in \Re_+^n : Ax = b\}.$$

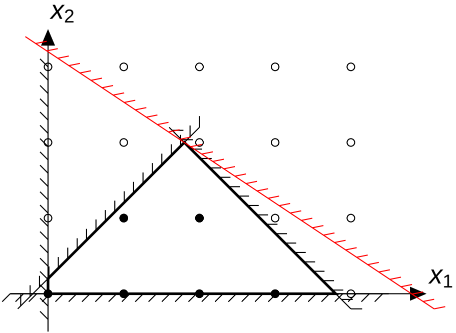
Goal: strengthening the formulation.

Idea: shift each constraint towards $conv(X)$ until it encounters an integer point.

Geometric approach (Chvátal)

A supporting hyperplane of a set S in a Euclidean space $\Re^n$ is a hyperplane such that

- S is entirely contained in one of the two closed half-spaces bounded by the hyperplane;
- S has at least one boundary-point on the hyperplane.



Geometric approach (Chvátal)

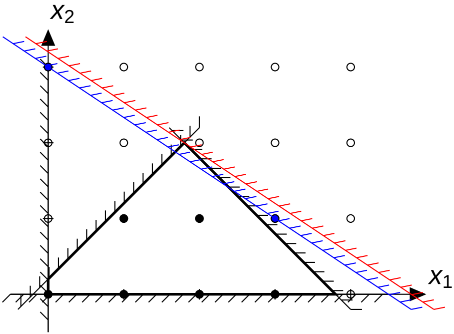
Let $\{x \in \Re^n_+ : h^T x = \theta\}$ be a supporting hyperplane of P with h integer.

Then $P \subseteq \{x \in \Re^n_+ : hx \leq \theta\}$.

Let $\Theta(P)$ the set of *all* supporting hyperplanes of P with integer lhs. Now consider a restriction $R(P)$ or P defined as follows:

$$R(P) := \bigcap_{(h,\theta) \in \Theta(P)} \{x \in \Re^n_+ : h^T x \leq \lfloor \theta \rfloor\}$$

Geometric approach (Chvátal)



Supporting hyperplane:

$$2x_1 + 3x_2 \leq 9.6$$

passing through vertex (1.8, 2.0).

Shifted hyperplane:

$$2x_1 + 3x_2 \leq 9$$

passing through integer points (0, 3), (3, 1)...

Geometric approach (Chvátal)

Although there may exist infinitely many supporting hyperplanes in $\Theta(P)$, it can be shown that $R(P)$ is still a polyhedron.

Obviously

$$conv(X) \subseteq R(P).$$

Therefore the process can be repeated iteratively, generating a sequence of polyhedra:

- $Q^0 = P$
- $Q^{t+1} = R(Q^t)\ \forall t \geq 0$

and

$$P = Q^0 \supseteq Q^1 \supseteq \ldots \supseteq conv(X).$$

Chvátal (1973) proved that $conv(X)$ is eventually obtained in a finite number of steps when P is a polytope.
Schrijvers (1980) proved that the result also applies to general polyhedra.

Algoritmic approach (Gomory)

Given a fractional solution x^* of the linear relaxation of an ILP problem, we strengthen the constraint associated with a fractional variable: we obtain a valid inequality violated by x^* and we iterate.

Given a discrete optimization problem

$$P) \quad \max\{cx : Ax = b, x \geq 0, x \in \mathcal{Z}^n_+\}$$

and its continuous linear relaxation

$$LP) \quad \max\{cx : Ax = b, x \geq 0\}$$

let x^* and z^* be the optimal solution of LP and its value.

$$z^* = \bar{a}_{00} + \sum_{j \in N^*} \bar{a}_{0j} x^*_j$$

$$\begin{cases} x^*_{B^*i} + \sum_{j \in N^*} \bar{a}_{ij} x^*_j = \bar{a}_{i0} & \forall i = 1, \dots, m \\ x^* \geq 0 \end{cases} \tag{1}$$

where B^* and N^* are the set of indices of basic and non-basic variables in x^*.

Gomory cuts

If x^* is not integer, there exists at least one constraint $\hat{i}$ s.t. $\bar{a}_{\hat{i}0}$ is not integer. Applying Chvátal-Gomory procedure to it, we obtain:

$$x_{B^*\hat{i}} + \sum_{j \in N^*} \lfloor \bar{a}_{\hat{i}j} \rfloor x_j \leq \lfloor \bar{a}_{\hat{i}0} \rfloor.$$

Subtracting this inequality from the equality constraint

$$x^*_{B^*\hat{i}} + \sum_{j \in N^*} \bar{a}_{\hat{i}j} x^*_j = \bar{a}_{\hat{i}0}$$

we obtain the Gomory cut:

$$\sum_{j \in N^*} f_{\hat{i}j} x_j \geq f_{\hat{i}0}$$

where $f_{\hat{i}j} = \bar{a}_{\hat{i}j} - \lfloor \bar{a}_{\hat{i}j} \rfloor$ and $f_{\hat{i}0} = \bar{a}_{\hat{i}0} - \lfloor \bar{a}_{\hat{i}0} \rfloor$.

The slack variable associated with this new inequality is also integer.

Gomory cuts

Gomory (1969) proved that with a particular choice of the generating row Gomory cuts lead to a finite algorithm, i.e., after adding a finite number of inequalities, an integer optimal solution is found.

Extension to MILP

The extension of Gomory's technique to mixed-integer problems is not trivial.

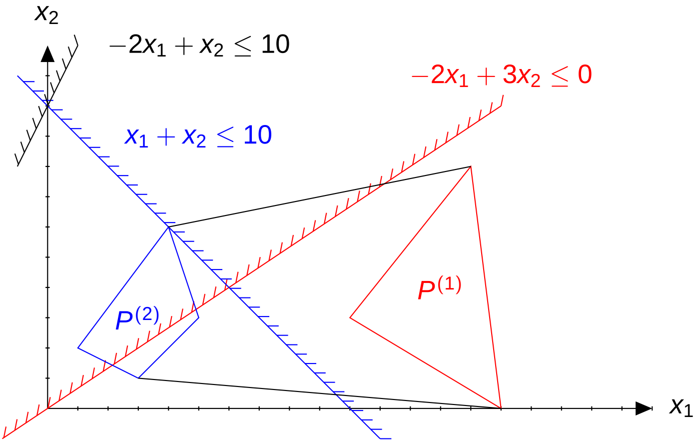
Observation. Let $u^{(1)}x \leq v^{(1)}$ and $u^{(2)}x \leq v^{(2)}$ be two valid inequalities, where the former one is valid for a polyhedron $P^{(1)}$ and the latter one for a polyhedron $P^{(2)}$. Then,

$$\sum_{j \in N} \min\{u_j^{(1)}, u_j^{(2)}\} x_j \leq \max\{v^{(1)}, v^{(2)}\}$$

is valid for $P^{(1)} \cup P^{(2)}$ and for $conv(P^{(1)} \cup P^{(2)})$.

Obviously, the same idea holds with reversed inequalities and swapping $\min$ and $\max$.

Geometric interpretation



Gomory mixed-integer cuts

Assume we have solved an LP to optimality.

We indicate by M the set of indices of the constraints (rows).

We indicate by B is the set of indices of basic variables (columns) and by N is the set of indices of non-basic variables (columns).

We indicate by $\overline{a}_{ij}$ the coefficient in the tableau on row i and column j.

We indicate by f_{ij} the fractional part of $\overline{a}_{ij}$, so that:

$$\overline{a}_{ij} = \alpha + f_{ij} \quad \text{for some integer } \alpha$$

Gomory mixed-integer cuts

Consider a basic variable x_k and the row i of the tableau where $\overline{a}_{ik} = 1$.

From linear programming we know that:

$$x_k = \overline{a}_{i0} - \sum_{j \in N} \overline{a}_{ij} x_j \quad \forall k \in B, \forall i \in M : \overline{a}_{ik} = 1.$$

The requirement

$$x_k \text{ integer}$$

is equivalent to the requirement that $\overline{a}_{i0}$ and $\sum_{j \in N} \overline{a}_{ij} x_j$ have the same fractional part, i.e.

$$\sum_{j \in N} \overline{a}_{ij} x_j = \alpha + f_{i0}$$

for some integer α.

Gomory mixed-integer cuts

We partition the set N into two subsets:

$$N_i^+ = \{j \in N : \overline{a}_{ij} \geq 0\}$$
$$N_i^- = \{j \in N : \overline{a}_{ij} < 0\}$$

and we consider two cases.

If $\sum_{j \in N} \overline{a}_{ij} x_j \geq 0$, then

$$\sum_{j \in N} \overline{a}_{ij} x_j = \alpha + f_{i0} \;\Rightarrow\; \sum_{j \in N_i^+} \overline{a}_{ij} x_j \geq f_{i0}.$$

If $\sum_{j \in N} \overline{a}_{ij} x_j < 0$, then

$$\sum_{j \in N} \overline{a}_{ij} x_j = \alpha + f_{i0} \;\Rightarrow\; \sum_{j \in N_i^-} \overline{a}_{ij} x_j \leq -1 + f_{i0}.$$

Gomory mixed-integer cuts

We define two polyhedra

$$P^{(1)} = \{x \in P : \sum_{j \in N} \bar{a}_{ij} x_j \geq 0\}$$

$$P^{(2)} = \{x \in P : \sum_{j \in N} \bar{a}_{ij} x_j < 0\}$$

For $P^{(1)}$ inequality $\sum_{j \in N_i^+} \bar{a}_{ij} x_j \geq f_{i0}$ is valid.
For $P^{(2)}$ inequality $\sum_{j \in N_i^-} \bar{a}_{ij} x_j \leq -1 + f_{i0}$ is valid.
We rewrite the latter one as $-\frac{f_{i0}}{1-f_{i0}} \sum_{j \in N_i^-} \bar{a}_{ij} x_j \geq f_{i0}$.

Now we use the observation above to obtain the valid inequality (mixed-integer Gomory cut)

$$\sum_{j \in N_i^+} \bar{a}_{ij} x_j - \frac{f_{i0}}{1 - f_{i0}} \sum_{j \in N_i^-} \bar{a}_{ij} x_j \geq f_{i0}.$$

Strengthening the cuts

We indicate by $N' \subseteq N$ the set of the integer variables.

The requirement

$$\left(\sum_{j \in N} \overline{a}_{ij} x_j\right) \mod 1 = f_{i0}$$

is equivalent to

$$\left(\sum_{j \in N \setminus N'} \overline{a}_{ij} x_j + \sum_{j \in N'} (\overline{a}_{ij} + k_{ij}) x_j\right) \mod 1 = f_{i0} \tag{2}$$

with k_{ij} integer $\forall j \in N'$.

Strengthening the cuts

Example:

$$\left(\frac{1}{10}x\right) \mod 1 = \frac{1}{2}$$

has the same integer solutions $x = 5, 15, 25, \ldots$ as

$$\left(\frac{11}{10}x\right) \mod 1 = \frac{1}{2}, \quad \left(\frac{21}{10}x\right) \mod 1 = \frac{1}{2}, \quad \ldots$$

and

$$\left(-\frac{9}{10}x\right) \mod 1 = \frac{1}{2}, \quad \left(-\frac{19}{10}x\right) \mod 1 = \frac{1}{2}, \quad \ldots$$

Strengthening the cuts

Therefore we can optimally select an integer k_{ij} for each $j \in N'$ to modify the coefficients of the integer variables in the resulting cut in order to make the cut as strong as possible.

The cut in $\geq$ form is made stronger, when the coefficients in its left-hand-side are minimized.

By adding or subtracting a suitable integer k_{ij} we can make the coefficient of x_j in (2) positive or negative, so that we can put j in N^+ or in N^-.

We indicate with a^*_{ij} the modified coefficient of x_j in (2):

$$a^*_{ij} = \overline{a}_{ij} + k_{ij}.$$

Now we examine two possible cases.

Strengthening the cuts

Case 1: $a^*_{ij} \geq 0$. Since j remains in N^+ or enters N^+, the coefficient of x_j in the cut left-hand-side is a^*_{ij}.

$$\begin{aligned} \text{minimize } a^*_{ij} = & \overline{a}_{ij} + k_{ij} \\ \text{s.t. } & \overline{a}_{ij} + k_{ij} \geq 0 \\ & k_{ij} \text{ integer} \end{aligned}$$

which is equivalent to

$$\begin{aligned} \text{minimize } & k_{ij} \\ \text{s.t. } & k_{ij} \geq -\overline{a}_{ij} \\ & k_{ij} \text{ integer} \end{aligned}$$

whose optimal solution is $k^*_{ij} = \lceil -\overline{a}_{ij} \rceil = -\lfloor \overline{a}_{ij} \rfloor$,
yielding a cut coefficient $a^*_{ij} = \overline{a}_{ij} + k_{ij} = \overline{a}_{ij} - \lfloor \overline{a}_{ij} \rfloor = f_{ij}$.

Strengthening the cuts

Case 2: $a^*_{ij} < 0$. Since j remains in N^- or goes in N^-, the coefficient of x_j in the cut left-hand-side is $\frac{f_{i0}}{1-f_{i0}}(-a^*_{ij})$.

$$\begin{aligned} \text{minimize } & \frac{f_{i0}}{1-f_{i0}}(-(\overline{a}_{ij} + k_{ij})) \\ \text{s.t. } & \overline{a}_{ij} + k_{ij} < 0 \\ & k_{ij} \text{ integer} \end{aligned}$$

which is equivalent to

$$\begin{aligned} \text{maximize } & k_{ij} \\ \text{s.t. } & k_{ij} < -\overline{a}_{ij} \\ & k_{ij} \text{ integer} \end{aligned}$$

whose optimal solution is $k^*_{ij} = \lceil -\overline{a}_{ij} \rceil - 1 = -\lfloor \overline{a}_{ij} \rfloor - 1$, yielding a cut coefficient $\frac{f_{i0}}{1-f_{i0}}(-a^*_{ij}) = \frac{f_{i0}}{1-f_{i0}}(-\overline{a}_{ij} + \lfloor \overline{a}_{ij} \rfloor + 1) = \frac{f_{i0}}{1-f_{i0}}(1 - f_{ij})$.

Strengthening the cuts

Finally, the choice between f_{ij} and $\frac{f_{i0}}{(1-f_{i0})}(1-f_{ij})$ is trivial: the former is the minimum when $f_{ij} \leq f_{i0}$, the latter when $f_{ij} \geq f_{i0}$.

So, the strengthened form of a mixed-integer Gomory cut is

$$\sum_{j \in N': f_{ij} \leq f_{i0}} f_{ij} x_j \quad + \sum_{j \in N': f_{ij} > f_{i0}} \frac{f_{i0}}{1-f_{i0}}(1-f_{ij}) x_j + $$
$$+ \sum_{j \notin N', j \in N_i^+} \bar{a}_{ij} x_j \quad + \sum_{j \notin N', j \in N_i^-} \frac{f_{i0}}{1-f_{i0}}(-\bar{a}_{ij}) x_j \geq f_{i0}.$$

Mixed-integer rounding cuts

Consider a mixed-integer problem with just two variables, one discrete and one continuous:

$$X = \{(x, y) \in \mathcal{Z} \times \Re_+ : x - y \leq b\}$$

where $b \in \Re$.

Nemhauser and Wolsey (1988) proved that

$$x - \frac{1}{1 - f(b)} y \leq \lfloor b \rfloor$$

is valid for X, where $f(b)$ indicates the fractional part of b.

Mixed-integer rounding cuts

Proof. Consider these two polyhedra:

$$P^{(1)} = \{(x, y) \in X : x \leq \lfloor b \rfloor\}$$

$$P^{(2)} = \{(x, y) \in X : x \geq \lfloor b \rfloor + 1\}$$

Clearly

$$X = P^{(1)} \cup P^{(2)}$$

Furthermore, rewrite the inequality

$$x - \frac{1}{1 - f(b)} y \leq \lfloor b \rfloor$$

as the equivalent inequality

$$(x - \lfloor b \rfloor)(1 - f(b)) \leq y.$$

This inequality is valid for both $P^{(1)}$ and $P^{(2)}$ and hence for X.

Mixed-integer rounding cuts

Consider $P^{(1)} = \{(x, y) \in X : x \leq \lfloor b \rfloor\}$.
The following two inequalities hold:

$$\begin{cases} x - \lfloor b \rfloor & \leq 0 \\ 0 & \leq y \end{cases}$$

Combining these inequalities with non-negative weights $1 - f(b)$ and 1, one obtains

$$(x - \lfloor b \rfloor)\,(1 - f(b)) \leq y.$$

Mixed-integer rounding cuts

Consider $P^{(2)} = \{(x, y) \in X : x \geq \lfloor b \rfloor + 1\}$.
The following two inequalities hold:

$$\begin{cases} -(x - \lfloor b \rfloor) & \leq -1 \\ x - y & \leq b \end{cases}$$

Combining these inequalities with non-negative weights $f(b)$ and 1, one obtains

$-x\,f(b) + \lfloor b \rfloor\,f(b) + x - y \leq -f(b) + b$
$-x\,f(b) + \lfloor b \rfloor\,f(b) + x - y \leq \lfloor b \rfloor$
$-x\,f(b) + \lfloor b \rfloor\,f(b) + x - \lfloor b \rfloor \leq y$
$x\,(1 - f(b)) - \lfloor b \rfloor\,(1 - f(b)) \leq y$

$$(x - \lfloor b \rfloor)\,(1 - f(b)) \leq y.$$

Mixed-integer rounding cuts

Now we consider a more general MILP with several discrete variables and a single continuous variable:

$$X = \{(x, y) \in \mathcal{Z}_+^n \times \Re_+ : ax - y \leq b\}$$

where $a \in \Re^n$ and $b \in \Re$.

Nemhauser and Wolsey (1988) proved that the following Mixed-Integer Rounding (MIR) inequality

$$\sum_{j=1}^{n} \left(\lfloor a_j \rfloor + \frac{(f(a_j) - f(b))^+}{1 - f(b)} \right) x_j - \frac{1}{1 - f(b)} y \leq \lfloor b \rfloor.$$

is valid for X, where $(\cdot)^+$ indicates $\max\{0, \cdot\}$.

Mixed-integer rounding cuts

We define $N^{(1)} = \{j \in \{1, \ldots, n\} : f(a_j) \leq f(b)\}$ and $N^{(2)} = N \backslash N^{(1)}$.

Since all variables are non-negative, then

$$\lfloor a_j \rfloor \, x_j \leq a_j \, x_j \;\; \forall j \in N \tag{3}$$

When $f(a_j) > 0$, we also have

$$a_j = \lceil a_j \rceil - (1 - f(a_j)) \;\; \forall j \in N : f(a_j) > 0 \tag{4}$$

We use inequality (3) for the terms in $N^{(1)}$ and equality (4) for the terms in $N^{(2)}$, so that $ax - y \leq b$ implies

$$\sum_{j \in N^{(1)}} \lfloor a_j \rfloor x_j + \sum_{j \in N^{(2)}} \lceil a_j \rceil x_j - \sum_{j \in N^{(2)}} (1 - f(a_j)) \, x_j - y \leq b.$$

Mixed-integer rounding cuts

$$\sum_{j \in N^{(1)}} \lfloor a_j \rfloor x_j + \sum_{j \in N^{(2)}} \lceil a_j \rceil x_j - \sum_{j \in N^{(2)}} (1 - f(a_j)) x_j - y \leq b.$$

We define

$$w = \sum_{j \in N^{(1)}} \lfloor a_j \rfloor x_j + \sum_{j \in N^{(2)}} \lceil a_j \rceil x_j$$

$$z = y + \sum_{j \in N^{(2)}} (1 - f(a_j)) x_j$$

and we observe that $z \geq 0$.

Therefore we can use the previous result to prove that

$$w - \frac{z}{1 - f(b)} \leq \lfloor b \rfloor$$

that corresponds to the MIR inequality (by substitution).

Lift-and-project cuts

Idea: represent the original polyhedron in a higher dimensional space (lifting), to obtain a valid inequality in the lifted space; then, project it back to the original space.

It only applies to mixed 0-1 linear programming.

Consider a mixed 0-1 linear program

$$\min\{cx : x \in X\} \quad X = \{x \in \{0,1\}^p \times \Re^{n-p} : Ax \leq b\}$$

where $Ax \leq b$ includes the bounds $0 \leq x_i \leq 1 \ \forall i = 1, \ldots, p$.

Let $P = \{x \in \Re^n : Ax \leq b\}$ be the polyhedron of the linear relaxation and $P_I = conv(X)$ the ideal formulation.

Lift-and-project cuts

An iteration of the lift-and-project algorithm (Balas, Céria, Cornuéjols, 1993) is made by three steps:

1. Select $j \in \{1, \ldots, p\}$.
2. Generate the two valid inequalities:

$$\begin{cases} (Ax)\, x_j \leq b\, x_j \\ (Ax)\,(1 - x_j) \leq b\,(1 - x_j) \end{cases}$$

   and replace $y_i = x_i\, x_j \;\; \forall i \neq j$ and $x_j = x_j^2$. Let $L_j(P)$ the lifted polyhedron obtained in this way.
3. Project $L_j(P)$ back by eliminating the y variables. Let P_j be the polyhedron obtained. The j^{th} component of each vertex of P_j is binary.

Lift-and-project cuts

Repeating the iteration for all j produces P_I, independently of the order.

To cut off the current fractional optimal solution x^* of the linear relaxation with a lift-and-project cut, it is sufficient to select j in step 1 so that x_j^* is fractional.

Step 3 requires to eliminate the y variables, i.e. to project a polyhedron

$$L_j(P) = \{(x, y) : Dx + By \leq d\}$$

into a polyhedron

$$P_j = \{x : (u^T D)x \leq u^T d, \forall u \in C\}$$

where $C = \{u : u^T B = 0, u \geq 0\}$ is a polyhedral cone.

Validity of the cuts

In contrast to the pure integer case, none of the cutting plane procedures presented (mixed-integer Gomory cuts, MIR, Lift-and-project cuts) yields a finite algorithm for MILP.

Adding Gomory cuts allows to reach the optimal solution of a MILP in a finite number of steps but this guarantee holds only if the values of the objective function are integer in all feasible solutions.

MIR inequalities provide a complete description of the polyhedron for any mixed 0-1 polyhedron.

Also Lift-and-project provides a finite algorithm only for mixed 0-1 programs.