Gomory cuts

Operational Research Complements

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Chvátal-Gomory procedure

Consider an IP problem with feasible set

$$X = \{x \in \mathcal{Z}^n_+ : Ax \leq b\}$$

where A has m rows and n columns.
Choose a vector $u \in \Re^m_+$:

- $\sum_{j=1}^n ua_j x_j \leq ub$ is valid because $ax \leq b$ and $u \geq 0$.
- $\sum_{j=1}^n \lfloor ua_j \rfloor x_j \leq ub$ is valid because $x \geq 0$.
- $\sum_{j=1}^n \lfloor ua_j \rfloor x_j \leq \lfloor ub \rfloor$ is valid because x is integer.

Any valid inequality can be generated with this procedure in a finite number of steps.

The effectiveness of the procedure depends on the choice of u.

Gomory cuts

Given a fractional solution x^* of the linear relaxation of a discrete optimization problem, we apply Chvátal-Gomory procedure to the constraint associated with a fractional variable: we obtain a valid inequality violated by x^* and we iterate.

Given a discrete optimization problem

$$P) \quad \max\{cx : ax = b, x \geq 0, x \in \mathcal{Z}^n\}$$

and its continuous linear relaxation

$$LP) \quad \max\{cx : ax = b, x \geq 0\}$$

let x^* and z^* be the optimal solution of LP and its value.

$$z^* = \overline{a}_{00} + \sum_{j \in NB^*} \overline{a}_{0j} x_j^*$$

$$\begin{cases} x_{B^*i}^* + \sum_{j \in NB^*} \overline{a}_{ij} x_j^* = \overline{a}_{i0} & \forall i = 1, \dots, m \\ x^* \geq 0 \end{cases} \tag{1}$$

where B^* and NB^* are the set of indices of basic and non-basic variables in x^*.

Gomory cuts

If x^* is not integer, there exists at least one constraint $\hat{i}$ s.t. $\overline{a}_{\hat{i}0}$ is not integer.

Applying Chvátal-Gomory procedure to it, we obtain:

$$x_{B^*\hat{i}} + \sum_{j \in NB^*} \lfloor \overline{a}_{\hat{i}j} \rfloor x_j \leq \lfloor \overline{a}_{\hat{i}0} \rfloor.$$

Subtracting this inequality from the equality constraint

$$x^*_{B^*\hat{i}} + \sum_{j \in NB^*} \overline{a}_{\hat{i}j} x^*_j = \overline{a}_{\hat{i}0}$$

we obtain the Gomory cut:

$$\sum_{j \in NB^*} f_{\hat{i}j} x_j \geq f_{\hat{i}0}$$

where $f_{\hat{i}j} = \overline{a}_{\hat{i}j} - \lfloor \overline{a}_{\hat{i}j} \rfloor$ and $f_{\hat{i}0} = \overline{a}_{\hat{i}0} - \lfloor \overline{a}_{\hat{i}0} \rfloor$.

The slack variable associated with this new inequality is also integer.

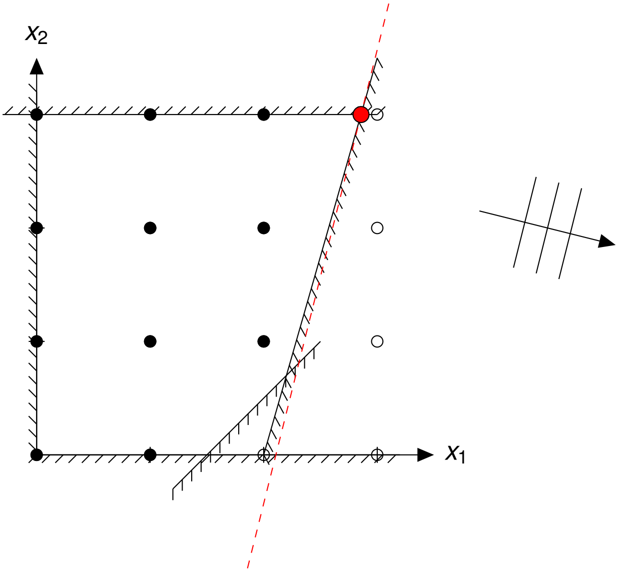
An example

$$
\begin{aligned}
\text{maximize } z = 4x_1 - x_2 & \\
7x_1 - 2x_2 &\leq 14 \\
x_2 &\leq 3 \\
2x_1 - 2x_2 &\leq 3 \\
x \geq 0 \text{ (integer)} &
\end{aligned}
$$

Solving the linear relaxation, we obtain $B^* = \{1, 2, 5\}$, $NB^* = \{3, 4\}$:

$$
\begin{aligned}
z = \frac{59}{7} \quad & - \frac{4}{7}x_3 - \frac{1}{7}x_4 \\
x_1 \quad & + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7} \\
x_2 \quad & + x_4 = 3 \\
& - \frac{2}{7}x_3 + \frac{10}{7}x_4 + x_5 = \frac{23}{7} \\
x \geq 0 &
\end{aligned}
$$

An example



An example

$$
\begin{aligned}
z = \frac{59}{7} \qquad & -\frac{4}{7}x_3 - \frac{1}{7}x_4 \\
x_1 \qquad & +\frac{1}{7}x_3 + \frac{2}{7}x_4 \qquad = \frac{20}{7} \\
x_2 \qquad & \qquad\quad +x_4 \qquad = 3 \\
& -\frac{2}{7}x_3 + \frac{10}{7}x_4 + x_5 = \frac{23}{7} \\
x \geq 0 &
\end{aligned}
$$

From the first constraint we can generate a Gomory cut:

$$x_1^* = \frac{20}{7} \Rightarrow \frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}.$$

Its slack variable is

$$s_1 = -\frac{6}{7} + \frac{1}{7}x_3 + \frac{2}{7}x_4.$$

An example

From the constraints

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7}$$
$$x_2 + x_4 = 3$$

we obtain

$$x_3 = -7x_1 + 2x_2 + 14$$
$$x_4 = -x_2 + 3$$

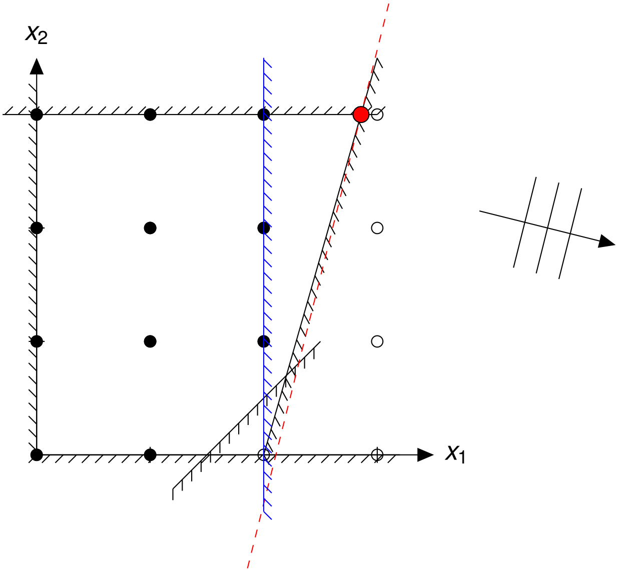
and the equation of the Gomory cut

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

can be rewritten as

$$x_1 \leq 2.$$

An example

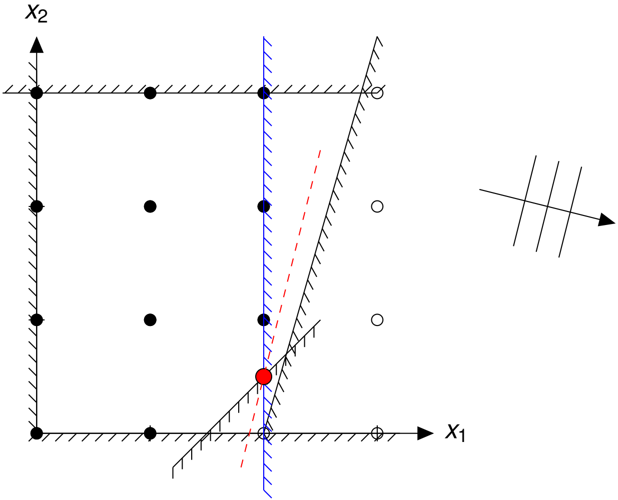


An example

Re-optimizing we obtain:

$$
\begin{aligned}
z = \frac{15}{2} \qquad\qquad & -\frac{1}{2}x_5 - 3s_1 \\
x_1 \qquad\qquad\qquad & \qquad\quad + s_1 = 2 \\
x_2 \qquad\quad & -\frac{1}{2}x_5 + \; s_1 = \frac{1}{2} \\
x_3 \quad & -x_5 - 5s_1 = 1 \\
& x_4 + \frac{1}{2}x_5 - s_1 = \frac{5}{2} \\
x, s \geq 0 &
\end{aligned}
$$

An example



An example

$$
\begin{aligned}
z = \frac{15}{2} \qquad\qquad & -\frac{1}{2}x_5 - 3s_1 \\
x_1 \qquad\qquad & \qquad\qquad +s_1 = 2 \\
x_2 \qquad & -\frac{1}{2}x_5 + \quad s_1 = \frac{1}{2} \\
x_3 \qquad & -x_5 - 5s_1 = 1 \\
& x_4 + \frac{1}{2}x_5 - s_1 = \frac{5}{2} \\
x, s \geq 0 &
\end{aligned}
$$

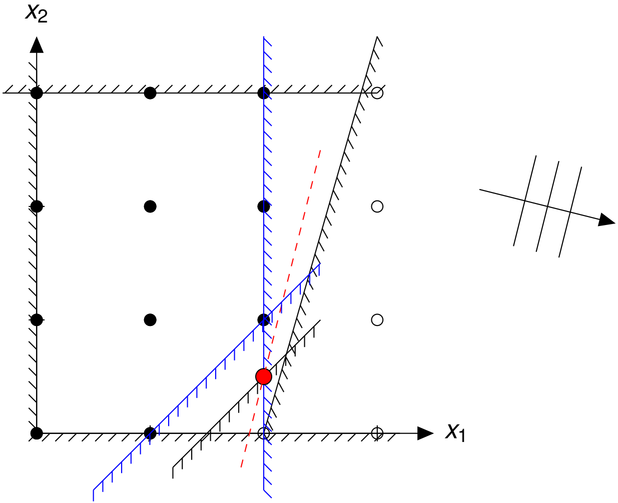
From the second constraint we can generate another Gomory cut:

$$x_2^* = \frac{1}{2} \quad \Rightarrow \quad \frac{1}{2}x_5 \geq \frac{1}{2} \quad \Rightarrow \quad x_1 - x_2 \leq 1.$$

Its slack variable is

$$s_2 = -\frac{1}{2} + \frac{1}{2}x_5.$$

An example



An example

Re-optimizing again we obtain:

$$
\begin{aligned}
z = 7 \quad & & -3s_1 - s_2 & \\
x_1 \quad & & +s_1 \quad & = 2 \\
x_2 \quad & & + s_1 - s_2 & = 1 \\
x_3 \quad & & -5s_1 - 2s_2 & = 2 \\
x_4 \quad & & -s_1 + s_2 & = 2 \\
x_5 \quad & & -2s_2 & = 1 \\
x, s \geq 0 & & &
\end{aligned}
$$

Now the optimal solution is integer.

An example