Bounds and relaxations

Giovanni Righini

Università degli Studi di Milano

Operations Research Complements

Università degli Studi
di Milano

Discrete optimization

Discrete optimization problems are in general very hard to solve, because:

- the number of solutions grows exponentially with the number of variables;
- mathematical tools such as derivatives (useful to characterize optima) are not available.

Owing to the *combinatorial explosion* in the number of solutions, explicit enumeration is not an option in general.

However, implicit enumeration algorithms can be used:

- branch-and-bound;
- dynamic programming.

Optimality

Given a discrete optimization problem P

$$z^* = \max\{z(x) : x \in \mathcal{X} \subseteq \mathcal{Z}^n\}$$

we prove optimality by computing an *upper bound* $\overline{z}$ and a *lower bound* $\underline{z}$, s.t.

$$\underline{z} \leq z^* \leq \overline{z}.$$

- If P is a minimization problem, $\overline{z}$ is a primal bound and $\underline{z}$ is a dual bound.
- If P is a maximization problem, $\underline{z}$ is a primal bound and $\overline{z}$ is a dual bound.

The difference $\overline{z} - \underline{z}$ is the *optimality gap*. When $\overline{z} - \underline{z} = 0$ we have an *optimality guarantee*.

Primal bounds

A primal bound $\overline{z}$ is given by the value of the objective function $z(x)$ in any feasible solution $\overline{x} \in \mathcal{X}$.

$$\overline{z} = z(\overline{x}) \;\; \overline{x} \in \mathcal{X}.$$

Primal bounds can be computed in many different ways:

- with heuristic and meta-heuristic algorithms (local search, GRASP,...);
- with approximation algorithms: in this case the algorithms provide both a primal and a dual bound.

For some discrete optimization problems it is computationally difficult to find a feasible solution (i.e. to compute primal bounds).

Dual bounds

A dual bound is given by the value of the objective function $z(x)$ in a super-optimal solution $\overline{x}$. Therefore in general $\overline{x}$ is not feasible.

There two main techniques to compute dual bounds for a problem P:

- to solve a relaxation R of P to optimality;
- to find a feasible solution to the dual D of P.

Relaxations

Given a problem

$$P = \min\{z_P(x) : x \in X(P)\}$$

a problem

$$R = \min\{z_R(x) : x \in X(R)\}$$

is a relaxation of P if the following two conditions hold:

- $X(P) \subseteq X(R)$
- $z_R(x) \leq z_P(x) \;\; \forall x \in X(P)$.

[In case of maximization, all inequalities must be reversed.]

Corollary: $z_R^* \leq z_P^*$.

There are many different types of relaxations.
A relaxation is better (tighter) than another when its optimal value is closer to z_P^*.

Continuous relaxation

When P is a discrete optimization problem

$$P) \min\{z(x) : x \in X, x \in \mathcal{Z}_+^n\},$$

its *continuous relaxation CR* is obtained from P by disregarding the integrality conditions:

$$CR) \min\{z(x) : x \in X, x \in \Re_+^n\}.$$

When P is a discrete *linear* optimization problem

$$P) \min\{cx : Ax \leq b, x \in \mathcal{Z}_+^n\},$$

its continuous relaxation

$$CR) \min\{cx : Ax \leq b, x \in \Re_+^n\}$$

is a linear programming problem (that can be solved very effectively).

If $x_{CR}^* \in \mathcal{Z}_+^n$, then $x_P^* = x_{CR}^*$.

Combinatorial relaxation

The combinatorial relaxation C of a combinatorial optimization problem P is still a combinatorial optimization problem, but typically much easier to solve.

Example 1:

- P: the asymmetric TSP;
- C: the min cost bipartite matching problem.

Example 2:

- P: the symmetric TSP;
- C: the min cost spanning 1-tree problem.

Lagrangean relaxation

The Lagrangean relaxation *LR* of a (discrete linear) optimization problem *P* is obtained by removing some constraints and adding penalties for their violations to the objective function.

$$P) \quad \min\{z(x) : Ax \leq b, x \in X \subseteq \mathcal{Z}_+^n\}$$

$$LR) \quad \min\{z_{LR}(x,\lambda) = z(x) + \lambda(Ax - b) : x \in X \subseteq \mathcal{Z}_+^n\}$$

with $\lambda \geq 0$.

It satisfies the two conditions for being a relaxation:

- Condition on the constraints: $\{x : Ax \leq b, x \in X\} \subseteq \{x : x \in X\}$
- Condition on the obj. function:
  - $Ax - b \leq 0$ for all feasible solutions of P;
  - $\lambda(Ax - b) \leq 0$ for all feasible solutions of P;
  - $z_{LR}(x,\lambda) = z(x) + \lambda(Ax - b) \leq z(x)$ for all feasible solutions of P.

Surrogate relaxation

The surrogate relaxation S of a (discrete linear) optimization problem P is obtained by replacing a set of constraints with their convex combination.

$$P) \quad \min\{z(x) : Ax \leq b, x \in X \subseteq \mathcal{Z}_+^n\}$$

$$S) \quad \min\{z(x) : \lambda^T Ax \leq \lambda^T b, x \in X \subseteq \mathcal{Z}_+^n\}$$

with $\lambda \geq 0$.

It satisfies the two conditions for being a relaxation:

- Condition on the constraints: $Ax \leq b$ implies $\lambda^T Ax \leq \lambda^T b$ but not viceversa.
- Condition on the obj. function: trivial, because it is the same.

Comparison

Linear, Lagrangean and surrogate relaxation can provide different bounds. Assuming minimization, the following relation holds:

$$z_{CR}^* \leq z_{LR}^*(\lambda^*) \leq z_S^*(\lambda^*) \leq z^*$$

Lagrangean and surrogate relaxation may provide tighter dual bounds than linear programming, provided that a suitable vector of multipliers λ^* is computed.

Duality

The second technique to obtain dual bounds is to find feasible solutions to the dual problem of P or the dual of a relaxation of P.

Linear dual problem:

$$P)z^* = \min\{cx : Ax \geq b, x \in \mathcal{Z}^n_+\}$$

$$D)w^* = \max\{yb : yA \leq c, y \in \Re^m_+\}$$

form a weak primal-dual pair.

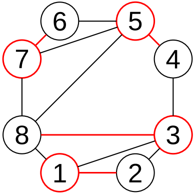
Combinatorial dual problem:
The maximum matching problem and minimum vertex cover problem

$$P)z^* = \max\{1x : Ax \leq 1, x \in \mathcal{B}^{|E|}_+\}$$

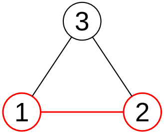
$$D)w^* = \min\{1y : yA \geq 1, y \in \mathcal{B}^{|V|}_+\}$$

where A is the incidence matrix of a graph $G = (V, E)$, form a weak primal-dual pair.

Example



$z^* = 4 \quad w^* = 4$



$$z^* = 1$$

$$z^*_{LP} = z([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]) = \frac{3}{2}$$

$$w^*_{LP} = w([\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]) = \frac{3}{2}$$

$$w^* = 2$$