Linear programming (recall)

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Linear programming (LP)

An optimization problem is a linear programming problem when:

- all its variables have a continuous domain;
- all its constraints are linear equations and inequalities;
- its objective function is a linear function of the variables.

In its general form al LP problem is:

$$\text{maximize/minimize } z = cx \tag{1}$$
$$\text{subject to } A_1 x \geq b_1 \tag{2}$$
$$A_2 x \leq b_2 \tag{3}$$
$$A_3 x = b_3 \tag{4}$$
$$x' \geq 0 \tag{5}$$
$$x'' \text{ unrestricted} \tag{6}$$

Contraints can be $\leq$, $\geq$ or $=$.
Some variables can be restricted to be non-negative.

Inequalities form

LP problems can be reformulated in an inequalities for, which is useful to give a geometrical interpretaation of the model.

To transform an LP model from its general form to an inequalites form, we must eliminate equality constraints and unrestricted variables, by substitution.

$$
\begin{array}{lrl}
maximize \;\; z = & c^T x & \\
s.t. & Ax & \leq b \\
 & x & \geq 0
\end{array}
$$

Geometrical interpretation of LP: constraints

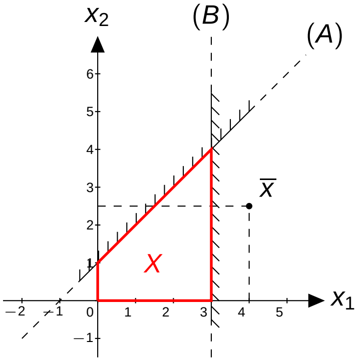
Every solution x is an assigment of values to the variables. Hence, it corresponds to a point in a continuous n-dimensional space, where n is the number of variables in the model.

Every equality constraint $ax = b$ corresponds to a hyperplane.
Every inequality constraint $ax \leq b$ corresponds to a halfspace.

The constraints system in the inequality form corresponds to the intersection of the corresponding halfspaces.
The intersection of halfspaces is a polyhedron.

Halfspaces are convex.
The intersection of convex sets is a convex set.
Hence, polyhedra are convex.

Geometrical interpretation of LP: constraints



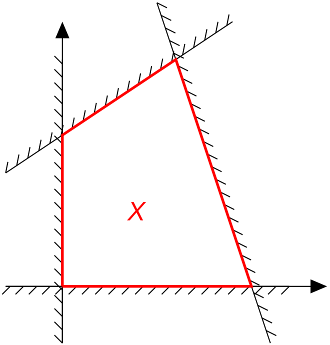
$$n = 2$$

$$\overline{x} = \begin{bmatrix} 4 \\ 2.5 \end{bmatrix}$$

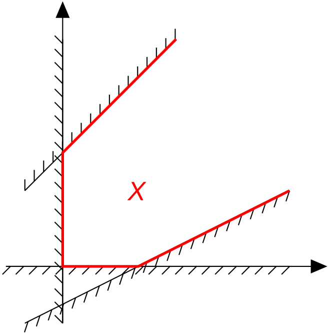
Feasible region:

$$X = \begin{cases} -x_1 + x_2 & \leq 1 \quad (A) \\ x_1 & \leq 3 \quad (B) \\ x \geq 0 & \end{cases}$$

Geometrical interpretation of LP: constraints

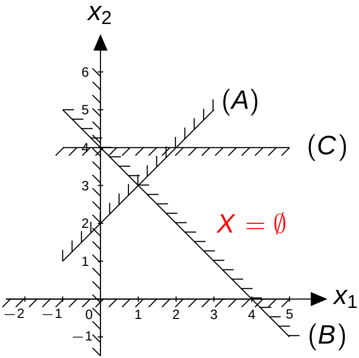


Bounded polyhedron (polytope)



Unbounded polyhedron

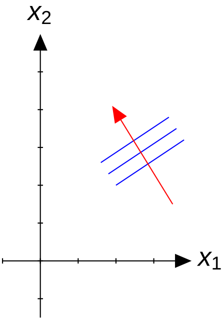
Geometrical interpretation of LP: constraints



Empty polyhedron

$$X = \begin{cases} -x_1 + x_2 \leq 2 & (A) \\ x_1 + x_2 \leq 4 & (B) \\ x_2 \geq 4 & (C) \\ x \geq 0 & \end{cases}$$

Geometrical interpretation of LP: objective



minimize $z = 2x_1 - 3x_2$

Since the objective function is linear, equivalent solutions lie on a same hyperplane.

The objective function is represented by a bundle of parallel hyperplanes, sorted according to the objective value.

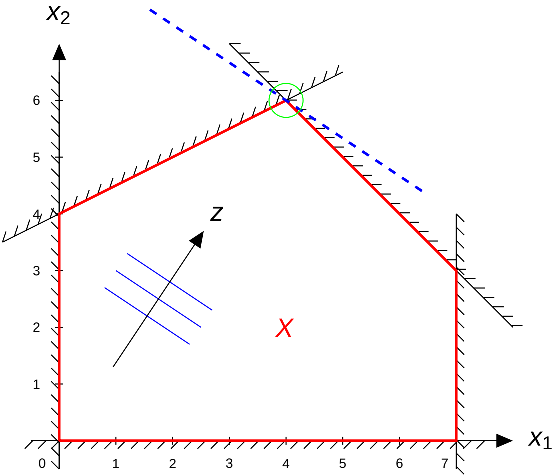
The optimization direction (minimization or maximization) defines the ordering of the hyperplanes.

Geometrical interpretation of LP

For the convexity of the polyhedron representing the feasible region and for the linearity of the indifference curves of the objective function (hyperplanes), only three cases may occur:

- the polyhedron is empty: no feasible solutions exist;
- the polyhedron is unbounded in the optimization direction: there is no finite optimal value;
- There exists at least a vertex of the polyhedron that corresponds to the optimal value.

Geometrical interpretation of LP



Standard form of LP

In an LP in standard form,

- the objective function is put in minimization form;
- all inequality constraints are rewritten as equations, by introducing suitable non-negative *slack* or *surplus* variables.

$$\begin{array}{ll} minimize \;\; z = & c^T x \\ s.t. & Ax + I\hat{x} = b \\ & x, \hat{x} \geq 0. \end{array}$$

Standard form

When rewriting an LP model in stadard form, starting from an inequalities form with m constraints and n variables, we obtain an LP model with m constraints and $n + m$ non-negative variables.

The constraints system is a system with m linear equations in $n + m$ variables.
If there are no redundant constraints, the rank of the coefficient matrix is m.
Hence, the system has a unique solution (if one exists) when we elimminate n degrees of freedom in excess, fixing n variables.

For each null variable in the standard form, there is an active constraint in the inequalities form.

Fixing n variables to 0 in the standard form corresponds to selecting a point in which n constraints are active in the inequalities form.

Base solutions

A base is a subset of m variables selected among the $n+m$ variables of the standard form.

$$[B \mid N]$$

The number of bases is combinatorial: it grows exponentially with m and n.

Once selected a base, the constraint system can be rewritten as follows:

$$Bx_B + Nx_N = b.$$

The solution of the $m \times m$ system that is obtained after fixing to 0 all the n non-basic variables is a base solution.
To obtain it, we must compute the inverse of the square submatrix B made by the basic columns:

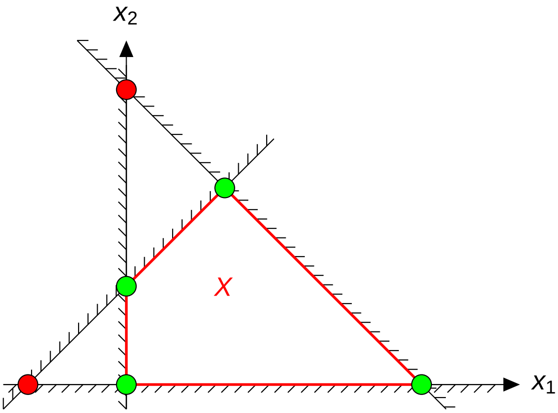
$$x_B = B^{-1}b - B^{-1}Nx_N$$

and then

$$x_N = 0 \quad x_B = B^{-1}b.$$

Base solutions

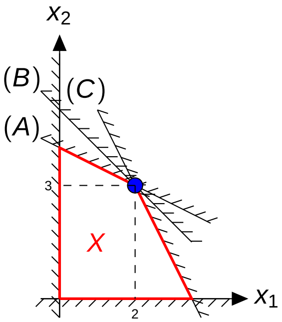
All the vertices of the polyhedron are base solutions but not vice versa: infeasible base solutions may exist (when $x_B \not\geq 0$).



Degeneracy

When a basic variable has null value, degeneracy occurs: two or more base solutions coincide.

In other words, more than n constraints are active in the same point in an n-dimensional space.



$$
\begin{array}{llrrll}
\textit{minimize} & z = & -x_1 & -x_2 & & \\
s.t. & & x_1 & +2x_2 & \leq 8 & (A) \\
 & & x_1 & +x_2 & \leq 5 & (B) \\
 & & 2x_1 & +x_2 & \leq 7 & (C) \\
 & & x_1, & x_2 & \geq 0 &
\end{array}
$$

The solution $x = [2\ 3\ 0\ 0\ 0]$ corresponds to the bases $\{1,2,3\}, \{1,2,4\}, \{1,2,5\}$.

Fundamental theorem of LP

Given a LP model in standard form,

$$z = \min\{c^T x : Ax = b, x \geq 0\}$$

with $rank(A) = m$

- if there exists a feasible solution,
  then there exists a feasible base solution;
- if there exists an optimal solution,
  then there exists an optimal base solution.

Therefore, a linear programming problem in a continuous space can be solved as a combinatorial problem, in a (discrete) space, by considering only base solutions.

Linear Programming (LP) duality

Every linear program P has a dual linear program D.

$$
\begin{aligned}
P)\ \min z = & \; c'^T x' + c''^T x'' \\
\text{s.t. } & a'x' + a''x'' \geq b' && [y'] \\
& d'x' + d''x'' = b'' && [y''] \\
& x' \geq 0 \\
& x'' \text{ free}
\end{aligned}
$$

$$
\begin{aligned}
D)\ \max w = & \; b'^T y' + b''^T y'' \\
\text{s.t. } & a'^T y' + d'^T y'' \leq c' && [x'] \\
& a''^T y' + d''^T y'' = c'' && [x''] \\
& y' \geq 0 \\
& y'' \text{ free}
\end{aligned}
$$

The fundamental theorem of LP duality

Given a primal-dual pair, one of these four cases occurs (and the simplex algorithm detects it in a finite number of steps):

- both P and D have a finite optimal solution;
- P is unbounded and D is infeasible;
- D is unbounded and P is infeasible;
- both P and D are infeasible.

Weak duality theorem

Weak duality theorem.
For each feasible solution x of P and for each feasible solution y of D, $z(x) \geq w(y)$.

Corollary 1.
If P is unbounded, then D is infeasible.

Corollary 2.
If x is feasible for P and y is feasible for D and $z(x) = w(y)$, then both x and y are also optimal.

Strong duality theorem

Strong duality theorem.
If there exist a feasible and optimal solution x^* for P, then there exists a feasible and optimal solution y^* for D and $z(x^*) = w(y^*)$.

Complementary slackness theorem

Complementary slackness theorem.
Given a feasible solution x for P and a feasible solution y for D, necessary and sufficient condition for them to be optimal is:

- Primal complementary slackness conditions (they are n):

$$x' \left(c' - a'^T y' - d'^T y''\right) = 0$$

- Dual complementary slackness conditions (they are m):

$$y' \left(a'x' + a''x'' - b'\right) = 0.$$

They only refer to inequality constraints, because those corresponding to equality constraints are always trivially satisfied.