The Held and Karp lower bound for the TSP

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The TSP

The Traveling Salesman Problem (TSP) is the problem of finding the minimum cost Hamiltonian cycle in an edge-weighted undirected graph.

Data. A graph $G = (V, E)$ and a weight function $c : E \mapsto \Re$ (wlog we can assume $c \geq 0$).

Variables. A binary variable x_e for each edge $e \in E$.

Objective. Minimize the total cost of the selected edges: $\sum_{e \in E} c_e x_e$.

The TSP

The constraints can be stated in several forms, providing different lower bounds when the continuous linear relaxation is solved.

A possibility is to impose:

- Degree constraints: every vertex must have degree equal to 2.

$$\sum_{e \in \delta(i)} x_e = 2 \ \ \forall i \in V$$

  where $\delta(i)$ indicates the subset of edges with an endpoint in vertex $i \in V$.

- Connectivity constraints: no subtours are allowed.

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \ \ \forall S \subseteq V \backslash \{r\}, S \neq \emptyset$$

  where $E(S)$ indicates the subset of edges in the subgraph induced by S and vertex $r \in V$ is chosen arbitrarily.

Lower bounding the TSP

The TSP is *NP*-hard.

Even its linear relaxation can be hard to compute, because of the exponential number of subtour elimination constraints

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \ \ \forall S \subseteq V \backslash \{r\}, S \neq \emptyset$$

that are needed to impose the connectivity of the solution.

Held and Karp (1970): procedure to compute the same lower bound that could be achieved from the linear relaxation, without the need of solving an LP with an exponential number of constraints.

Idea: solve a suitable Lagrangean relaxation of the TSP, such that the subproblem requires to compute an optimal connected subset of edges that can be found with little computational effort: a 1-tree.

Degree constraints

The degree constraints $\sum_{e\in\delta(i)} x_e = 2 \ \ \forall i \in V$ can be replaced by an equivalent reformulation

$$\begin{cases} \sum_{e\in\delta(i)} x_e = 2 & \forall i \in V \backslash \{r\} \\ \sum_{e\in V} x_e = n \end{cases}$$

where $r \in V$ is arbitrarily selected.

Proof. The degree constraints are explicitly imposed $\forall i \neq r$. We must prove that the reformulation implies $\sum_{e\in\delta(i)} x_e = 2$ also for $i = r$.

$$\sum_{e\in E} x_e = \frac{1}{2}\sum_{i\in V}\sum_{e\in\delta(i)} x_e = \frac{1}{2}\sum_{i\in V\backslash\{r\}}\sum_{e\in\delta(i)} x_e + \frac{1}{2}\sum_{e\in\delta(r)} x_e = (n-1) + \frac{1}{2}\sum_{e\in\delta(r)} x_e.$$

Hence $\sum_{e\in E} x_e = n$ implies $\sum_{e\in\delta(r)} x_e = 2$. $\square$

The reformulated TSP

We obtain the following reformulation for the TSP.

$$
\begin{aligned}
\text{minimize } z = & \sum_{e \in E} c_e x_e \\
\text{s.t. } & \sum_{e \in \delta(i)} x_e = 2 & \forall i \in V \setminus \{r\} \\
& \sum_{e \in E} x_e = n \\
& \sum_{e \in E(S)} x_e \leq |S| - 1 & \forall S \subseteq V \setminus \{r\}, S \neq \emptyset \\
& x_e \in \{0, 1\} & \forall e \in E
\end{aligned}
$$

Now we can compute the Lagrangean relaxation of the degree constraints for all vertices but one.

The Lagrangean relaxation

The Lagrangean relaxation is the following problem.

$$\text{minimize } z_{LR}(x,\lambda) = \sum_{e \in E} c_e x_e + \sum_{i \in V \setminus \{r\}} \lambda_i \Big(\sum_{e \in \delta(i)} x_e - 2 \Big)$$

$$\text{s.t. } \sum_{e \in V} x_e = n \qquad (1)$$

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \qquad \forall S \subseteq V \setminus \{r\}, S \neq \emptyset \qquad (2)$$

$$x_e \in \{0,1\} \qquad \forall e \in E \qquad (3)$$

Multipliers λ are unrestricted in sign.
Constraints (1)-(3) define spanning 1-trees.
Constraints (3) can be relaxed (integrality property).

The Lagrangean objective function

In the Lagrangean subproblem the objective is:

$$\text{minimize } z_{LR}(x, \lambda)) = \sum_{e \in E} c_e x_e + \sum_{i \in V \setminus \{r\}} \lambda_i \Big(\sum_{e \in \delta(i)} x_e - 2 \Big)$$

Introducing $\lambda_r = 0$ for ease of notation, the Lagrangean objective function can be rewritten as

$$\text{minimize } z_{LR}(x, \lambda)) = \sum_{[i,j] \in E} (c_{ij} + \lambda_i + \lambda_j) x_{ij} - 2 \sum_{i \in V} \lambda_i$$

The Lagrangean sub-problem can be restated on a graph $G'(\lambda)$ with modified costs $c'_{ij} = c_{ij} + \lambda_i + \lambda_j \quad \forall [i,j] \in E$:

$$\text{minimize } z_{LR}(x, \lambda)) = \sum_{[i,j] \in E} c'_{ij} x_{ij} - 2 \sum_{i \in V} \lambda_i .$$

The modified costs graph

In graph $G'(\lambda)$ the cost of all Hamiltonian tours is increased by the same constant term $2\sum_{i\in V}\lambda_i$.

Therefore their ranking remains unchanged.

In graph $G'(\lambda)$ the cost $T(x)$ of any spanning 1-tree is modified as follows:

$$T'(x,\lambda) = T(x) + \sum_{i\in V} d_i(x)\lambda_i - 2\sum_{i\in V}\lambda_i,$$

where $d_i(x)$ indicates the degree of each vertex $i \in V$.

The Lagrangean lower bound

In $G'(\lambda)$

- the optimal tour of cost C^* remains optimal with cost

$$C'^*(\lambda) = C^* + 2\sum_{i\in V} \lambda_i$$

- the minimum cost spanning 1-tree provides a valid lower bound to the cost of the optimal Hamiltonian cycle:

$$T'^*(\lambda) \leq C'^*(\lambda).$$

Therefore

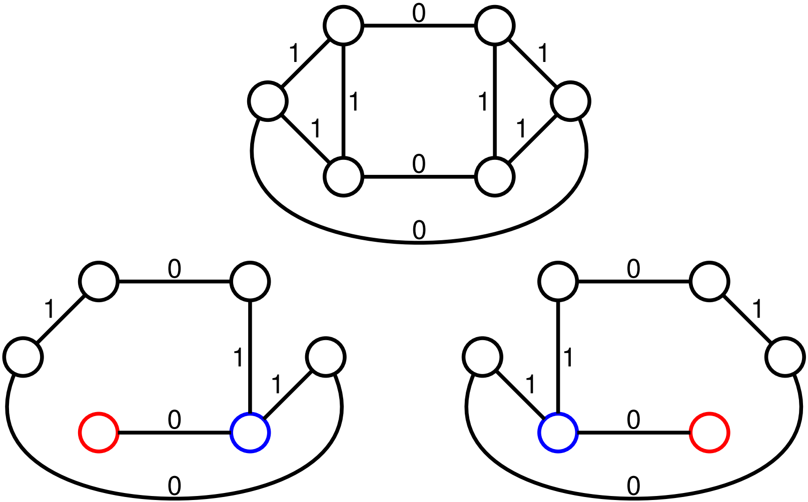
$$C^* = C'^*(\lambda) - 2\sum_{i\in V} \lambda_i \geq T'^*(\lambda) - 2\sum_{i\in V} \lambda_i = \min_x \{z_{LR}(x, \lambda)\} = z^*_{LR}(\lambda).$$

Integrality gap

In general the gap between C^* and $z^*_{LR}(\lambda)$ cannot be closed.

Since the Lagrangean subproblem has the integrality property, the best lower bound achievable in this way is the same lower bound that would be obtained from linear relaxation, but it is obtained in a more practical way, especially for large graphs.

Example



A convex combination of these two 1-trees gives $LB = 3$, but $C^* = 4$.

Computing a minimum cost spanning 1-tree

A spanning 1-tree is made by [two equivalent definitions]

- a tree spanning V and an additional edge;
- a tree spanning $V \setminus \{r\}$ and two additional edges incident to vertex r for some arbitrary choice of $r \in V$.

Held and Karp used the second definition, with $r = 1$.

A minimum cost spanning tree can be computed in polynomial time and so is a minimum cost spanning 1-tree.

The Lagrangean dual problem

For any choice of λ, $z^*_{LR}(\lambda)$ is a valid lower bound of C^*.

To use the lower bound within a branch-and-bound algorithm, we are interested in its tighest (i.e. maximum) value.

The Lagrangean dual problem is to maximize $z^*_{LR}(\lambda)$ with a suitable choice of λ.

For this purpose it is possible to use subgradient optimization (Held and Karp (1970, 1971), Held, Wolfe and Crowder (1974)) with improvements suggested by Helbig Hansen and Krarup (1974), Smith and Thompson (1977), Jonker and Volgenant (1982).

Subgradient optimization

The multipliers λ are iteratively updated according to the following rule:

$$\lambda_i^{(k+1)} \leftarrow \lambda_i^{(k)} + t^{(k)}(d^{(k)}(i) - 2) \quad \forall i \in V,$$

where $d^{(k)}(i)$ is the degree of vertex $i \in V$ in the minimum cost 1-tree computed at each iteration k.

If $d^{(k)}(i) > 2$, the potential of vertex i is increased;
if $d^{(k)}(i) < 2$, the potential of vertex i is decreased;
if $d^{(k)}(i) = 2$, the potential of vertex i is left unchanged.

The step t is updated with the heuristic rule:

$$t^{(k)} \leftarrow \frac{\pi^{(k)}(UB - z_{LR}^*(\lambda^{(k)}))}{\sum_{i \in V}(d^{(k)}(i) - 2)^2},$$

where UB is an upper bound and $\pi^{(k)}$ is a scalar decreasing with k.

Lower bound 1

Lower bound 1. A minimum cost 1-tree is a valid lower bound.

It is made by

- the minimum spanning tree T^*;
- the minimum cost edge among those not in T^*.

However, this is not the largest valid lower bound that can be achieved from 1-trees.

Lower bounds 2 and 3

Lower bound 2 (Held and Karp, 1970). A valid lower bound can be obtained by

- the minimum spanning tree T_r^* computed after the deletion of vertex r;
- the two minimum cost edges with an endpoint in r.

It holds $LB2 \geq LB1$ because $LB2$ is the cost of a 1-tree, while $LB1$ is the cost of a minimum cost 1-tree.

Lower bound 3 (Helsgaun). A valid lower bound can be obtained by

- the minimum spanning tree T^*;
- the minimum cost edge $[i, j]$ incident to a leaf i of T^* and not in T^*.

It holds $LB3 \geq LB1$ because $LB3$ is the cost of a 1-tree, while $LB1$ is the cost of a minimum cost 1-tree.

Lower bounds 2 and 3

Both $LB2$ and $LB3$ depend on the selection of a vertex:

- $LB2$ depends on the vertex r which is selected not to be included in the spanning tree;
- $LB3$ depends on the leaf i of T^* which is selected.

Then, both $LB2$ and $LB3$ can be maximized by selecting r or i in an optimal way.

Let $LB2^* = \max_{r \in V}\{LB2(r)\}$ and $LB3^* = \max_{i \in Leafs(T^*)}\{LB3(i)\}$.

There is no guaranteed dominance between $LB2^*$ and $LB3^*$.