Lagrangean relaxation

Giovanni Righini

Operational Research Complements

UNIVERSITÀ DEGLI STUDI DI MILANO

Relaxations

Given a problem $\mathcal{P}$, such as:

$$\begin{aligned} \text{minimize } & z_{\mathcal{P}}(x) \\ \text{s.t. } & x \in X_{\mathcal{P}} \end{aligned}$$

a problem $\mathcal{R}$, such as:

$$\begin{aligned} \text{minimize } & z_{\mathcal{R}}(x) \\ \text{s.t. } & x \in X_{\mathcal{R}} \end{aligned}$$

is a relaxation of $\mathcal{P}$ if and only if these two conditions hold:

- $X_{\mathcal{P}} \subseteq X_{\mathcal{R}}$
- $z_{\mathcal{R}}(x) \leq z_{\mathcal{P}}(x) \;\; \forall x \in X_{\mathcal{P}}$.

As a consequence

$$z^*_{\mathcal{R}} = z_{\mathcal{R}}(x^*_{\mathcal{R}}) \leq z_{\mathcal{R}}(x^*_{\mathcal{P}}) \leq z_{\mathcal{P}}(x^*_{\mathcal{P}}) \leq z^*_{\mathcal{P}}.$$

The optimal value of the relaxation is not worse than the optimal value of the original problem.

Lagrangean relaxation

Consider this problem $\mathcal{P}$:

$$\begin{aligned} \mathcal{P}) \text{ minimize } z_{\mathcal{P}}(x) = c^T x & \\ \text{s.t. } A_1 x \geq b_1 & \\ A_2 x \geq b_2 & \\ x \geq 0 \text{ integer} & \end{aligned}$$

A **Lagrangean relaxation** of $\mathcal{P}$ is:

$$\begin{aligned} \mathcal{LR}) \text{ minimize } z_{\mathcal{LR}}(x, \lambda) = & c^T x + \lambda^T (b_1 - A_1 x) \\ \text{s.t. } & A_2 x \geq b_2 \\ & x \geq 0 \text{ integer} \end{aligned}$$

Violations of the relaxed constraints are penalized in the objective function by means of Lagrangean multipliers $\lambda \geq 0$.

Is it a relaxation?

$$\mathcal{P}) \min z_{\mathcal{P}}(x) = c^T x \quad \text{s.t. } A_1 x \geq b_1,\; A_2 x \geq b_2,\; x \geq 0 \text{ integer}$$

$$\mathcal{LR}) \min z_{\mathcal{LR}}(x, \lambda) = c^T x + \lambda^T (b_1 - A_1 x) \quad \text{s.t. } A_2 x \geq b_2,\; x \geq 0 \text{ integer}$$

1. Feasible region:

$$X_{\mathcal{P}} \subseteq X_{\mathcal{LR}}$$

   because constraints $A_1 x \geq b_1$ have been relaxed.

2. Objective function:

$$\left.\begin{array}{r} \forall x \in X_{\mathcal{P}} \Rightarrow A_1 x \geq b_1 \\ \forall \lambda \geq 0 \end{array}\right\} \Rightarrow \lambda^T (b_1 - A_1 x) \leq 0 \Rightarrow z_{\mathcal{LR}}(x, \lambda) \leq z_{\mathcal{P}}(x).$$

Variations

Lagrangean relaxation can also be applied to equality constraints. In this case λ is unrestricted in sign.

Lagrangean relaxation can also be applied to non-linear constraints.

Lagrangean dual problem

For each choice of the Lagrangean multipliers $\lambda \geq 0$ we obtain a different instance, providing a different dual bound.

Example (Set Covering).

The Lagrangean dual problem asks for the optimal Lagrangean multipliers λ^*:

$$\mathcal{LD})\ z^*_{\mathcal{LD}} = \max_{\lambda \geq 0} \{ z^*_{\mathcal{LR}}(\lambda) = \min_{x \in \mathcal{Q}} \{ z_{\mathcal{LR}}(x, \lambda) = c^T x + \lambda^T (b_1 - A_1 x) \} \}.$$

This provides the best dual bound that can be obtained from the Lagrangean relaxation of constraints $A_1 x \geq b_1$. But a duality gap may still exist between $z^*_{\mathcal{LD}}$ and $z^*_{\mathcal{P}}$.

Complementary slackness

If the optimal solution x^* for problem $\mathcal{LR}$ is also feasible for $\mathcal{P}$, this does not guarantee that x^* is optimal for $\mathcal{P}$. This is true only if the penalty terms are null.

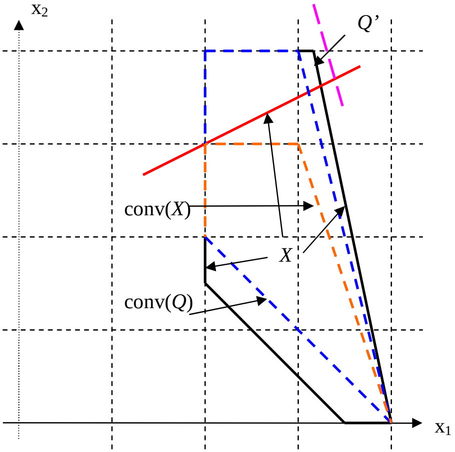
Primal optimality requires both:

- **feasibility**: $x^* \in X_{\mathcal{P}}$, i.e. $A_1 x^* \geq b_1$
- **complementarity**: $\lambda^{*T}(b_1 - A_1 x^*) = 0$.

When equality constraints are relaxed, the two conditions coincide.

Geometrical interpretation (1)

$$\begin{aligned}
\mathcal{P})\ \max z_{\mathcal{P}}(x) = & 7x_1 + 2x_2 \\
\text{s.t. } & {\color{red}-x_1 + 2x_2 \leq 4} \\
& 5x_1 + x_2 \leq 20 \\
& -2x_1 - 2x_2 \leq -7 \\
& -x_1 \leq -2 \\
& x_2 \leq 4 \\
& x \in \mathcal{Z}^2_+
\end{aligned}$$



Geometrical interpretation (2)

$\mathcal{LR})\ \max z_{\mathcal{LR}}(x, \lambda) =$

$= (7 + \lambda)x_1 + (2 - 2\lambda)x_2 + 4\lambda$

s.t. $5x_1 + x_2 \leq 20$

$-2x_1 - 2x_2 \leq -7$

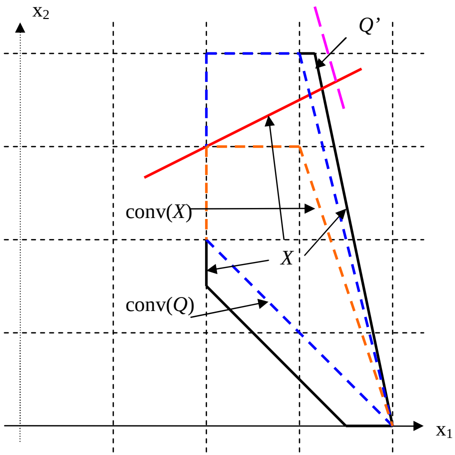
$-x_1 \leq -2$

$x_2 \leq 4$

$x \in \mathcal{Z}_+^2$

$\mathcal{Q} = \{(2,2), \ldots, (4,0)\}$.

$\mathcal{Q}'$ is the polyhedron defined by the linear constraints of $\mathcal{LR}$.



Primal viewpoint

The objective function $z_{\mathcal{LR}}(x, \lambda)$ can be seen as...

- a function of x for λ fixed;
- a function of λ for x fixed.

Primal viewpoint.

For each given $\overline{\lambda} \geq 0$ we have

$$z^*_{\mathcal{LR}}(\overline{\lambda}) = \max\{z_{\mathcal{LR}}(x, \overline{\lambda}) : x \in \mathcal{Q}\}.$$

Since constraints and objectives are linear:

$$z^*_{\mathcal{LR}}(\overline{\lambda}) = \max\{z_{\mathcal{LR}}(x, \overline{\lambda}) : x \in conv(\mathcal{Q})\}.$$

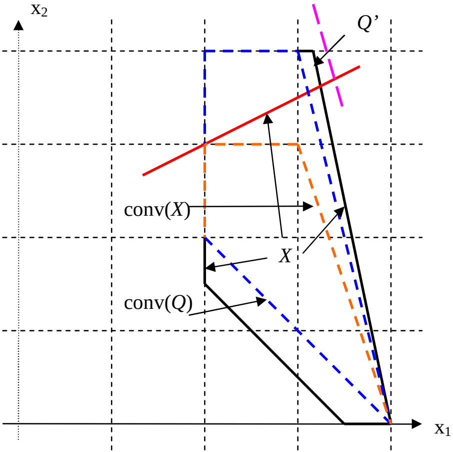
This is a linear programming problem, where the orientation of the objective function depends on $\overline{\lambda}$.

Primal viewpoint (example).

For $0 \le \overline{\lambda} \le \frac{1}{9} \rightarrow$
$\rightarrow z^*_{\mathcal{LR}}(\overline{\lambda}) = z((3,4), \overline{\lambda}) =$
$29 - \overline{\lambda}$.

For $\overline{\lambda} \ge \frac{1}{9} \rightarrow$
$\rightarrow z^*_{\mathcal{LR}}(\overline{\lambda}) = z((4,0), \overline{\lambda}) =$
$28 + 8\overline{\lambda}$.

Hence
$z^*_{\mathcal{LD}} = \min_{\overline{\lambda} \ge 0}\{z^*_{\mathcal{LR}}(\overline{\lambda})\} =$
$z^*_{\mathcal{LR}}(\frac{1}{9}) = z_{\mathcal{LR}}((3,4), \frac{1}{9}) =$
$z_{\mathcal{LR}}((4,0), \frac{1}{9}) = 28 + \frac{8}{9}$.



Dual viewpoint

For each given $x^i \in \mathcal{Q}$, $z_{\mathcal{LR}}(x^i, \lambda)$ is an affine function of λ.
Hence $z^*_{\mathcal{LR}}(\lambda) = \max\{z_{\mathcal{LR}}(x^i, \lambda) : x^i \in \mathcal{Q}, \lambda \geq 0\}$ is piecewise linear and convex.

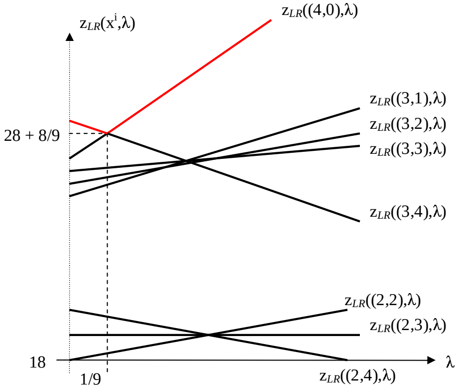
$z^*_{\mathcal{LR}}(\lambda) = z_{\mathcal{LR}}((3,4), \lambda)$
for $0 \leq \lambda \leq \frac{1}{9}$

$z^*_{\mathcal{LR}}(\lambda) = z_{\mathcal{LR}}((4,0), \lambda)$
for $\lambda \geq \frac{1}{9}$.

Then

$$z^*_{\mathcal{LD}} = \min\{z^*_{\mathcal{LR}}(\lambda) : \lambda \geq 0\} =$$

$$= z^*_{\mathcal{LR}}(\frac{1}{9}) = 28 + \frac{8}{9}.$$



In our example when $\lambda = \frac{1}{9}$ we have $z^*_{\mathcal{LR}}(\lambda) = 28 + \frac{8}{9}$.

Consider the point $x^* = \frac{8}{9}(3,4) + \frac{1}{9}(4,0)$, which is a convex combination of two integer points of $\mathcal{Q}$. Point x^* is the intersection point between the segment joining the two integer points and the relaxed constraint.

$$28 + \frac{8}{9} = z_{\mathcal{LR}}((3,4), \frac{1}{9}) = z_{\mathcal{LR}}((4,0), \frac{1}{9}) = z_{\mathcal{LR}}(x^*, \frac{1}{9}) =$$

$$= c^T x^* + \frac{1}{9}(4 - (-x_1^* + 2x_2^*)) = c^T x^*.$$

In general:

$$z^*_{\mathcal{LD}} = \max\{c^T x : A_1 x \leq b_1, x \in conv(\mathcal{Q})\}.$$

There exists a convex combination x^* of points of $\mathcal{Q}$, which satisfies the relaxed constraints and such that the value of the objective function in x^* is equal to $z^*_{\mathcal{LD}}$. Finding this convex combination which satisfies the relaxed constraints and maximizes the original objective function is a linear problem and it is the dual of the Lagrangean dual.

How good is Lagrangean relaxation?

Let define $\mathcal{R} = \{x \in \Re^n_+ : A_1 x \leq b_1\}$.

$$conv(X_{\mathcal{P}}) = conv(\mathcal{Q} \cap \mathcal{R}) \subseteq conv(\mathcal{Q}) \cap \mathcal{R}$$

Hence: $z^*_{\mathcal{P}} = \max\{c^T x : x \in X_{\mathcal{P}}\} = \max\{c^T x : x \in conv(X_{\mathcal{P}})\} =$

$\max\{c^T x : x \in conv(\mathcal{Q} \cap \mathcal{R})\} \leq \max\{c^T x : x \in conv(\mathcal{Q}) \cap \mathcal{R}\} = z^*_{\mathcal{LD}}$.

The duality gap $z^*_{\mathcal{LD}} - z^*_{\mathcal{P}}$ depends on the relationship between $X_{\mathcal{P}}$ and $\mathcal{Q}$ (and then $\mathcal{R}$) and from the coefficients c.

$z^*_{\mathcal{LD}} = z^*_{\mathcal{P}}$ $\forall c$ if and only if $conv(\mathcal{Q} \cap \mathcal{R}) = conv(\mathcal{Q}) \cap \mathcal{R}$.

In our example they do not coincide: $z^*_{\mathcal{LD}} = 28 + \frac{8}{9}$ and $z^*_{\mathcal{P}} = 28$.

Lagrangean vs. linear relaxation

When the relaxed problem is a discrete one, it is interesting to compare the dual bound $z^*_{\mathcal{LD}}$ that can be obtained from Lagrangean relaxation with the dual bound $z^*_{\mathcal{LP}}$ which is (easily) obtained from linear relaxation.

$$z^*_{\mathcal{LD}} = z^*_{\mathcal{LP}} \ \ \forall c$$

if all extreme points of the polyhedron $\{x \in \Re^n_+ : A_2 x \leq b_2\}$ are integer, i.e. $\{x \in \Re^n_+ : A_2 x \leq b_2\} = conv(\mathcal{Q})$.

In our example this does not occur and we have

$$z^*_{\mathcal{LP}} = 30 + \frac{2}{11} > z^*_{\mathcal{LD}} = 28 + \frac{8}{9}.$$

If we relaxed all constraints but the last two, we would obtain a feasible region $\{x \in \Re^n_+ : A_2 x \leq b_2\}$ with integer extreme points. In that case we would have $z^*_{\mathcal{LP}} = z^*_{\mathcal{LD}} = 30 + \frac{2}{11}$.

Lagrangean vs. linear relaxation

Summing up:

$$conv(X_{\mathcal{P}}) \subseteq conv(\mathcal{Q}) \cap \mathcal{R} \subseteq \{x \in \Re^n_+ : Ax \leq b\}$$

and then

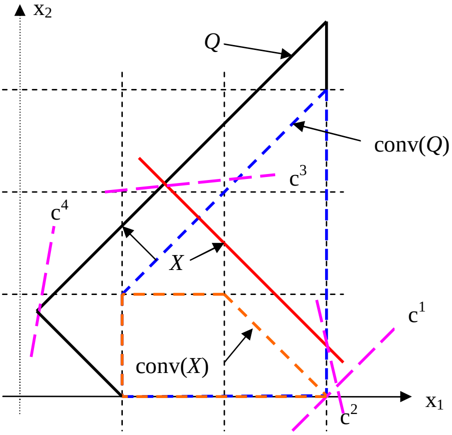
$$z^*_{\mathcal{P}} \leq z^*_{\mathcal{LD}} \leq z^*_{\mathcal{LP}}.$$

In general the best dual bound given by Lagrangean relaxation is tighter than the dual bound given by linear relaxation. They are equivalent when the relaxed problem has the integrality property (all integral extreme points).

Special cases

For some particular orientation of the objective function the bounds can be the same even if the general conditions for equivalence are not satisfied.

With c^1: $z^*_{\mathcal{P}} = z^*_{\mathcal{LD}} = z^*_{\mathcal{LP}}$.
With c^2: $z^*_{\mathcal{P}} < z^*_{\mathcal{LD}} = z^*_{\mathcal{LP}}$.
With c^3: $z^*_{\mathcal{P}} < z^*_{\mathcal{LD}} < z^*_{\mathcal{LP}}$.
With c^4: $z^*_{\mathcal{P}} = z^*_{\mathcal{LD}} < z^*_{\mathcal{LP}}$.



Observations

- In linear relaxation the objective function and the constraints remain unchanged; in Lagrangean relaxation they are modified.
- In linear relaxation the integrality restrictions are dropped; in Lagrangean relaxation they are kept.
- If the optimal solution of a linear relaxation is integer, then it is optimal for the original problem; if the optimal solution of a Lagrangean relaxation satisfies the relaxed constraints, this does not guarantee that it is optimal for the original problem.
- The linear relaxation of a problem is unique; Lagrangean relaxation can be defined relaxing different constraints.
- The value of a solution x in a linear relaxation only depends on x; the value of a solution x in a Lagrangean relaxation depends on λ, too.

Approximation properties

$$z^*_{\mathcal{LD}} - z^*_{\mathcal{P}} \leq \epsilon \Leftrightarrow \exists \overline{\lambda} \geq 0, \overline{x} \in X_{\mathcal{P}} :$$

- $\overline{\lambda}^T (b_1 - A_1 \overline{x}) \leq \delta_1$
- $z^*_{\mathcal{LR}}(\overline{\lambda}) - z_{\mathcal{LR}}(\overline{x}, \overline{\lambda}) \leq \delta_2$
- $\delta_1 + \delta_2 \leq \epsilon$.

Therefore we can have $z^*_{\mathcal{LD}} = z^*_{\mathcal{P}}$ only if there exist $\overline{\lambda} \geq 0$ and $\overline{x} \in X_{\mathcal{P}}$ such that these two conditions hold:

- **Optimality**: $z^*_{\mathcal{LR}}(\overline{\lambda}) = z_{\mathcal{LR}}(\overline{x}, \overline{\lambda})$
- **Complementarity**: $\overline{\lambda}^T (b_1 - A_1 overlinex) = 0$.

We can evaluate the distance from optimality if we know

- δ_1: the error due to the choice of $\overline{x}$ in the $\mathcal{LR}$;
- δ_2: the error due to the choice of $\overline{\lambda}$ in the Lagrangean dual problem.

Lagrangean relaxation and problem decomposition (1)

A typical application of Lagrangean relaxation is to relax coupling constraints: this allows to decompose the problem into independent sub-problems.

Example: Uncapacitated facility location problem.

$$\min z_{\mathcal{P}}(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} x_{ij} + \sum_{j=1}^{N} f_j y_j$$

$$\text{s.t.} \sum_{j=1}^{N} x_{ij} = (\geq) 1 \qquad \forall i = 1, \ldots, M$$

$$x_{ij} \leq y_j \qquad \forall i = 1, \ldots, M \ \forall j = 1, \ldots, N$$

$$y \in \mathcal{B}^n$$

Lagrangean relaxation and problem decomposition (2)

Relaxing the first set of constraints we have penalties of the form $\sum_{i=1}^{M} \lambda_i (1 - \sum_{j=1}^{N} x_{ij})$:

$$\min z_{\mathcal{LR}}(x, y, \lambda) = \sum_{i=1}^{M} \sum_{j=1}^{N} (c_{ij} - \lambda_i) x_{ij} + \sum_{j=1}^{N} f_j y_j + \sum_{i=1}^{M} \lambda_i$$

$$\text{s.t. } x_{ij} \le y_j \qquad \forall i = 1, \dots, M \ \forall j = 1, \dots, N$$

$$y \in \mathcal{B}^n$$

This can be decomposed into N independent subproblems:

$$\min z_j(x, y, \lambda) = \sum_{i=1}^{M} (c_{ij} - \lambda_i) x_{ij} + f_j y_j$$

$$\text{s.t. } x_{ij} \le y_j \qquad \forall i = 1, \dots, M$$

$$x_{ij} \ge 0 \qquad \forall i = 1, \dots, M$$

$$y_j \in \{0, 1\}$$

that can be easily solved.

Which relaxation should we choose?

There is a trade-off between three criteria:

- How many constraints do we relax? (n. of Lagrangean multipliers)
- How difficult the relaxed problem is? (computing time)
- Does the relaxed problem have the integrality property? (tightness of the dual bound)

An example

Consider for instance the Generalized Assignment Problem.

$$\min z_{\mathcal{P}}(x) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} x_{ij}$$

$$\text{s.t.} \sum_{j=1}^{N} x_{ij} = (\geq) 1 \qquad \forall i = 1, \ldots, M$$

$$\sum_{i=1}^{M} a_{ij} x_{ij} \leq b_j \qquad \forall j = 1, \ldots, N$$

$$x \in \mathcal{B}^{M \times N}$$

We can relax the assignnment constraints, the capacity constraints or both: we get three different relaxations.

Lagrangean relaxation 1

Penalty terms of constraints $\sum_{j=1}^{N} x_{ij} = (\geq) 1 \; \forall i = 1, \ldots, M$ have the form $\sum_{i=1}^{M} \lambda_i (1 - \sum_{j=1}^{N} x_{ij})$. Then we obtain:

$$\min z^1_{\mathcal{LR}}(x, \lambda) = \sum_{i=1}^{M} \sum_{j=1}^{N} (c_{ij} - \lambda_i) x_{ij} + \sum_{i=1}^{M} \lambda_i$$

$$\text{s.t.} \sum_{i=1}^{M} a_{ij} x_{ij} \leq b_j \qquad \forall j = 1, \ldots, N$$

$$x \in \mathcal{B}^{M \times N}$$

This relaxation:

- requires M multipliers,
- implies solving N knapsack problem instances,
- does not have the integrality property.

Lagrangean relaxation 2

Penalty terms of constraints $\sum_{i=1}^{M} a_{ij} x_{ij} \leq b_j \; \forall j = 1, \ldots, N$ have the form: $\sum_{j=1}^{N} \mu_j (\sum_{i=1}^{M} a_{ij} x_{ij} - b_j)$. Then we obtain:

$$\min z^2_{\mathcal{LR}}(x, \mu) = \sum_{i=1}^{M} \sum_{j=1}^{N} (c_{ij} + a_{ij} \mu_j) x_{ij} - \sum_{j=1}^{N} \mu_j b_j$$

$$\text{s.t.} \sum_{j=1}^{N} x_{ij} = (\geq) 1 \qquad \forall i = 1, \ldots, M$$

$$x \in \mathcal{B}^{M \times N}$$

This relaxation:

- requires N multipliers,
- implies solving M trivial problem instances,
- does have the integrality property.

Lagrangean relaxation 3

Relaxing both sets of constraints, we obtain:

$$\min z^3_{\mathcal{LR}}(x, \lambda, \mu) = \sum_{i=1}^{M} \sum_{j=1}^{N} (c_{ij} - \lambda_i + a_{ij}\mu_j) x_{ij} + \sum_{i=1}^{M} \lambda_i - \sum_{j=1}^{N} \mu_j b_j$$

$$\text{s.t. } x \in \mathcal{B}^{M \times N}$$

This relaxation:

- requires $M + N$ multipliers,
- implies solving $M \times N$ trivial problem instances,
- does have the integrality property.

How do we solve the Lagrangean dual problem?

The Lagrangean dual problem consists of minimizing a piecewise linear and convex function in a space with as many dimensions as the number of Lagrangean multipliers.

- **Convexity**: good news. There are no sub-optimal local minima.
- **Piecewise linearity**: bad news. First order partial derivatives (gradient vector) are not continuous everywhere.

It is a linear programming problem with as many constraints as the number of integer points in $\mathcal{Q}$.

To solve this problem we can use one of several approximation methods:

- sub-gradient optimization,
- multiplier adjustment,
- dual ascent.

Sub-gradient optimization (1)

It is a local search algorithm that iteratively moves from a current point $\lambda^{(k)}$ to a next point $\lambda^{(k+1)}$ with a step of length $\sigma^{(k)}$ in the direction opposite of the sub-gradient $b - Ax^*(\lambda^{(k)})$.

Begin
Inizialize $\lambda^{(0)}$; k:=0;
while not EndTest **do**
  Solve $\max z_{\mathcal{LR}}(x, \lambda^{(k)}) = c^T x + \lambda^{(k)T}(b - Ax)$ s.t. $x \in \mathcal{Q}$;
  Obtain $x^*(\lambda^{(k)})$;
  Choose $\sigma^{(k)}$;
  $\lambda^{(k+1)} := \max\{\lambda^{(k)} - \sigma^{(k)}(b - Ax^*(\lambda^{(k)})), 0\}$;
  $k := k + 1$;
end while
End

The sub-gradient algorithm is not monotonic: it is necessary to save the best value encountered so far.

Sub-gradient optimization (2)

- **Initialization** is usually not important. The algorithm is not sensitive to it.
- The **end test** succeeds when no improvements are observed after a given number of iterations.
- The **step** can be chosen with some heuristic rule.

Rule of thumb (a). If

- $\lim_{k\to\infty}\sum_{i=1}^{k}\sigma^{(i)}=\infty$ and
- $\lim_{k\to\infty}\sigma^{(k)}=0$,

then the sub-gradient algorithm converges to the optimal value of the Lagrangean dual.

Rule of thumb (b). If $\sigma^{(k)}=\sigma^{(0)}\rho^{k}$ with $\rho<1$, then it converges to optimality provided that $\mu^{(0)}$ and ρ are large enough.

Sub-gradient optimization (3)

Beasley's rules of thumb:

- Normalize the right hand sides of the relaxed constraints to 1;
- Choose $0 < \pi \leq 2$;
- Choose $\lambda^{(0)}$ arbitrarily;
- Define a scalar step $\sigma = \frac{\pi|\overline{z} - z^*_{\mathcal{LR}}(\lambda^{(0)}, x^*)|}{\sum_i \max\{G_i, 0\}^2}$, where
  - $\overline{z}$ is the value of a feasible solution (possibly increased by a small percentage);
  - x^* is the optimal solution of $\mathcal{LR}(\lambda^{(0)})$;
  - $z^*_{\mathcal{LR}}(\lambda^{(0)}, x^*)$ is the corresponding optimal value;
  - i is the index spanning all relaxed constraints;
  - $G_i = b_i - \sum_{j=1}^{n} a_{ij} x_j^*$ is the sub-gradient of constraint i evaluated in x^*;
- Update $\lambda_i := \max\{0, \lambda_i + \sigma G_i\}$;
- Halve π when the best value does not improve for a certain number of iterations;
- Stop after a prescribed number of iterations of when π becomes "small enough".

Algorithmic improvements

Lagrangean heuristics.

The information provided by the optimal solution of the Lagrangean relaxation (corresponding to a dual bound), is used to build a primal feasible solution.

Lagrangean heuristics must be tailored to each specific problem and relaxation.
This allows to compute primal bounds every time dual bounds are computed in a branch-and-bound algorithm.

Variable fixing.

After solving a Lagrangean relaxation, we have:

- a reduced cost for each variable,
- a gap between a primal and a dual bound.

If the reduced cost is larger than the gap, the current value of the variable can be fixed.

Lagrangean decomposition (1)

Given a problem

$$\begin{aligned} \min \;& c^T x \\ \text{s.t. } & A_1 x \geq b_1 \\ & A_2 x \geq b_2 \\ & x \in \mathcal{B}^n \end{aligned}$$

we can transform it into the equivalent problem

$$\begin{aligned} \min \;& c'^T x + (c - c')^T y \\ \text{s.t. } & A_1 x \geq b_1 \\ & A_2 y \geq b_2 \\ & x = y \\ & x, y \in \mathcal{B}^n \end{aligned}$$

Now we can relax the coupling constraint $x = y$ and decompose.

Lagrangean decomposition (2)

From the Lagrangean relaxation of the coupling constraints (equalities) we have:

$$\begin{aligned} \min\ & c'^T x + (c - c')^T y + \lambda^T (x - y) \\ \text{s.t. } & A_1 x \geq b_1 \\ & A_2 y \geq b_2 \\ & x, y \in \mathcal{B}^n \end{aligned}$$

which decomposes into

$$\begin{aligned} \min\ & (c' + \lambda)^T x \\ \text{s.t. } & A_1 x \geq b_1 \\ & x \in \mathcal{B}^n \end{aligned}$$

$$\begin{aligned} \min\ & (c - c' - \lambda)^T y \\ \text{s.t. } & A_2 y \geq b_2 \\ & y \in \mathcal{B}^n \end{aligned}$$