Programming in Python[1]

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Lecture XVII: Laplacian operator

Gray-Scott systems

Systems driven by the Gray-Scott's equation exhibit Turing patterns (D_u, D_v, f, k are constants).

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u - uv^2 + f \cdot (1 - u)$$
$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + uv^2 - (f + k) \cdot v$$

- These give the change of u and v chemicals over time
- The diffusion term can be approximated on a grid by computing the discrete Laplacian

Discrete Laplacian

$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

- Change on a grid (1-D):

$$\nabla f[n] = f[n+1] - f[n]$$
$$\nabla f[n] = f[n] - f[n-1]$$

- Second order change (1-D):

$$\begin{aligned}\nabla(\nabla f[n]) &= \nabla(f[n+1]) - \nabla(f[n]) \\ &= (f[n+1] - f[n]) - (f[n] - f[n-1]) \\ &= f[n-1] - 2f[n] + f[n+1]\end{aligned}$$

- In 2-D we do this independently on the 2 dimensions n, m:

$$\begin{aligned}\nabla(\nabla f[n,m]) &= f[n-1,m] - 2f[n,m] + f[n+1,m] + \\ &\quad f[n,m-1] - 2f[n,m] + f[n,m+1] \\ &= f[n-1,m] + f[n+1,m] + f[n,m-1] + f[n,m+1] - 4f[n,m]\end{aligned}$$

Vectorization

0	0	0	0	0	0
0	13	14	15	16	0
0	9	10	11	12	0
0	5	6	7	8	0
0	1	2	3	4	0
0	0	0	0	0	0

-29	-18	-19	-37
-8	0	0	-13
-4	0	0	9
3	2	1	-5

Same trick we used for "life", but we need to compute the *5-point stencil* with these weights (see previous derivation):

0	1	0
1	-4	1
0	1	0

This way one can compute the Laplacian matrix using only vectorized plus.

Vectorization

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`X[1:-1, 2:]`

Same trick we used for "life", but we need to compute the *5-point stencil* with these weights (see previous derivation):

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```
X[2:, 1:-1]
```

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`X[:-2, 1:-1]`

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0	0	0	0	0	0

`X[1:-1, 1:-1]`

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Consider also the diagonals

Another approximation which takes into account also the "diagonals" is the *9-point stencil*.

1	1	1
1	-8	1
1	1	1

Experimental evidence

Turing proposed his model on a pure theoretical basis, but we have now also some experimental evidence:

> *Economou, A. D., Ohazama, A., Porntaveetus, T., Sharpe, P. T., Kondo, S., Basson, M. A., Gritli-Linde, A., Cobourne, M. T., Green, J. B. (2012). Periodic stripe formation by a Turing mechanism operating at growth zones in the mammalian palate. Nature genetics, 44(3), 348–351.* https://doi.org/10.1038/ng.1090