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Lecture XVII: Laplacian operator

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Programming in Python¹

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Gray-Scott systems

Systems driven by the Gray-Scott's equation exhibit Turing **patterns** $(D_u, D_v, f, k \text{ are constants}).$

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u - u v^2 + f \cdot (1 - u)$$
$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + u v^2 - (f + k) \cdot v$$

- These give the **change** of *u* and *v* chemicals over time
- The diffusion term can be approximated on a grid by computing the discrete Laplacian

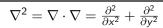


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Gray-Scott

Discrete Laplacian



Change on a grid (1-D):

$$\nabla f[n] = f[n+1] - f[n]$$
$$\nabla f[n] = f[n] - f[n-1]$$

Second order change (1-D):

$$\nabla(\nabla f[n]) = \nabla(f[n+1]) - \nabla(f[n])$$

$$= (f[n+1] - f[n]) - (f[n] - f[n-1])$$

$$= f[n-1] - 2f[n] + f[n+1]$$

• In 2-D we do this independently on the 2 dimensions n, m:

$$\nabla(\nabla f[n,m]) = f[n-1,m] - 2f[n,m] + f[n+1,m] + f[n,m-1] - 2f[n,m] + f[n,m+1]$$

$$= f[n-1,m] + f[n+1,m] + f[n,m-1] + f[n,m+1] - 4f[n,m]$$

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Discrete Laplacian

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Vectorization



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Discrete Laplacian

0	0	0	0	0	0
0	13	14	15	16	0
0	9	10	11	12	0
0	5	6	7	8	0
0	1	2	3	4	0
0	0	0	0	0	0

-29	-18	-19	-37
-8	0	0	-13
-4	0	0	9
3	2	1	-5

$$X[1:-1, 2:] X[2:, 1:-1]$$

$$X[1:-1, :-2]$$
 $X[:-2,$

1:-1] X[1:-1, 1:-1] Same trick we used for "life", but we need to compute the 5-point stencil with these weights (see previous derivation):

0	1	0
1	-4	1
0	1	0

This way one can compute the Laplacian matrix using only vectorized plus.

Experimental evidence



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Discrete Laplacian

Turing proposed his model on a pure theoretical basis, but we have now also some experimental evidence:

Economou, A. D., Ohazama, A., Porntaveetus, T., Sharpe, P. T., Kondo, S., Basson, M. A., Gritli-Linde, A., Cobourne, M. T., Green, J. B. (2012). Periodic stripe formation by a Turing mechanism operating at growth zones in the mammalian palate. Nature genetics, 44(3), 348-351. https://doi.org/10.1038/ ng. 1090

Consider also the diagonals



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Another approximation which takes into account also the "diagonals" is the 9-point stencil.

1	1	1
1	-8	1
1	1	1

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