Programming in Python[1]

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Lecture XXIV: Probabilistic programming

How science works

Describing one single “scientific method” is problematic, but a schema many will accept is:

1. Imagine a hypothesis
2. Design (mathematical/convenient) models consistent with the hypothesis
3. Collect experimental data
4. Discuss the fitness of data given the models

It is worth noting that the falsification of models is not *automatically* a rejection of hypotheses (and, more obviously, neither a validation).

The role of Bayes Theorem

In this discussion, a useful relationship between data and models is Bayes Theorem.

$$P(M, D) = P(M|D) \cdot P(D) = P(D|M) \cdot P(M)$$

Therefore:

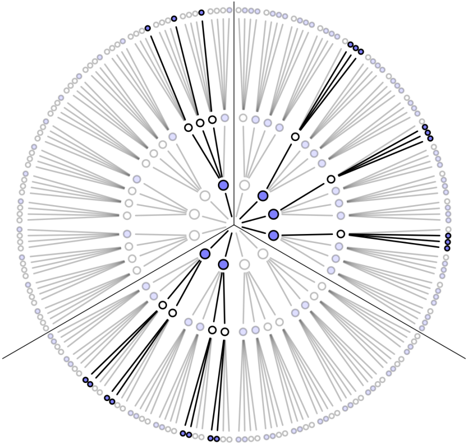
$$P(M|D) = \frac{P(D|M) \cdot P(M)}{P(D)}$$

The plausibility of the model given some observed data, is proportional to the number of ways data can be *produced* by the model and the prior plausibility of the model itself.

Simple example

- Model: a bag with 4 balls in 2 colors `B/W` (but we don't know which of `BBBB`, `BBBW`, `BBWW`, `BWWW`, `WWWW`)
- Observed: `BWB`
- Which is the plausibility of `BBBB`, `BBBW`, `BBWW`, `BWWW`, `WWWW`?

Bayes Theorem is counting



Picture from: R. McElreath, Statistical Rethinking

A computational approach

This Bayesian strategy is (conceptually) easy to transform in a computational process.

1. Code the models
2. Run the models
3. Compute the plausibility of the models based on observed data

Classical binomial example

- Which is the proportion p of water covering Earth? The models are indexed by the float $0 < p < 1$
- Given p, the probability of observing some `W`,`L` in a series of independent random observations is: $P(W, L|p) = \frac{(W+L)!}{W!\cdot L!} p^W \cdot (1-p)^L$ (binomial distribution).
- Do we have an initial (prior) idea?
- Make observations, apply Bayes, update prior!

A conventional way of expressing the model

$$
\begin{aligned}
W &\sim \mathit{Binomial}(W + L, p) \\
p &\sim \mathit{Uniform}(0, 1)
\end{aligned}
$$

Probabilistic programming is systematic way of coding this kind of models, combining predefined statistical distributions and Monte Carlo methods for computing the posterior plausibility of parameters.

In principle you can do it by hand


```python
def dbinom(success: int, size: int, prob: float) -> float:

    fail = size - success
    return math.factorial(size)/(math.factorial(success)*math.factorial(fail))*prob**succ
    ↪   ess*(1-prob)**(fail)
```

Then,

```python
W, L = 7, 3    # for example 'WWWLLWWLWW'
p_grid = np.linspace(start=0, stop=1, num=20)
prior = np.ones(20)/20

likelihood = dbinom(W, n=W+L, p=p_grid)

unstd_posterior = likelihood * prior

posterior = unstd_posterior / unstd_posterior.sum()
```

Unfeasible with many variables!

```
import pymc as pm

W, L = 7, 3
earth = pm.Model()
with earth:
    p = pm.Uniform("p", 0, 1)  # uniform prior
    w = pm.Binomial("w", n=W+L, p=p, observed=W)
    posterior =  pm.sample(2000)

posterior['p']
```