An ASP Approach to Generate Minimal Countermodels in Intuitionistic Propositional Logic

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Intuitionistic Propositional Logic

Intuitionistic Propositional Logic (IPL) is a constructive non-classical logic.

- Non-classical: some classical tautologies are not valid in IPL

$$A \vee \neg A \qquad (A \to B) \vee (B \to A) \qquad \neg A \vee (A \to B)$$

- Constructive: IPL enjoys the Disjunction Property:

$$A \vee B \in \mathrm{IPL} \quad \Longrightarrow \quad A \in \mathrm{IPL} \text{ or } B \in \mathrm{IPL}$$

IPL is closely related to Propositional Classical Logic (CPL):

- $\mathrm{IPL} \subset \mathrm{CPL}$
- IPL can be embedded in CPL:

$$A \in \mathrm{IPL} \quad \Longrightarrow \quad \neg\neg A \in \mathrm{IPL}$$

Thus, the following principles are valid in IPL:

$$\neg\neg(A \vee \neg A) \qquad \neg\neg((A \to B) \vee (B \to A)) \qquad \neg\neg(\neg A \vee (A \to B))$$

Semantics

- CPL

  An interpretation $\mathcal{I}$ is a set of propositional variables.

  The validity of a formula w.r.t. $\mathcal{I}$ is defined according to the classical meaning of logical connectives (truth tables):

  - $\mathcal{I} \models p$ iff $p \in \mathcal{I}$, for p a propositional variable
  - $\mathcal{I} \models A \wedge B$ iff $\mathcal{I} \models A$ and $\mathcal{I} \models B$
  - ...

  CPL is the set of formulas valid in all the interpretations.

- IPL

  To get a sound semantics for IPL, we need a more refined semantics. A model is a set of classical interpretations, called worlds

  - √ Each world represents a knowledge state
  - √ Worlds are ordered by a partial order relation $\leq$
  - √ Validity is represented by forcing relation $\Vdash$ between worlds and formulas
  - √ Forcing is preserved by $\leq$:

$$w_1 \Vdash A \quad \wedge \quad w_1 \leq w_2 \quad \Longrightarrow \quad w_2 \Vdash A$$

This leads to Kripke frame semantics.

Semantics

A Kripke model is a structure $\mathcal{K} = \langle P, \leq, V \rangle$, where:

- P is a finite nonempty set of worlds
- $\leq$ is a partial order between worlds
- V assign to each world a classical interpretation, obeying truth preservation:

$$w_1 \leq w_2 \implies \mathcal{I}(w_1) \subseteq \mathcal{I}(w_2)$$

- The forcing relation $\Vdash$ between worlds and formulas is inductively defined as follows:
  - $w \nVdash \bot$
  - $w \Vdash p$ iff $V(w) \models p$ ($V(w)$ is the interpretation related to w)
  - $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$
  - $w \Vdash A \vee B$ iff $\alpha \Vdash A$ or $w \Vdash B$
  - $w \Vdash \neg A$ iff, for every $w' \geq w$, $w' \nVdash A$
  - $w \Vdash A \rightarrow B$ iff, for every $w' \geq w$, $w' \nVdash A$ or $w' \Vdash B$

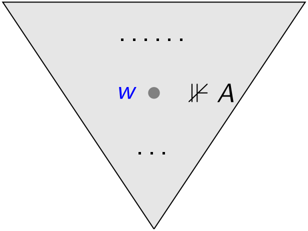
For formulas $\neg A$ and $A \rightarrow B$, forcing at w depends on the successors of w

Semantics

IPL is complete with respect to Kripke semantics, namely:

- $A \in \mathrm{IPL}$ iff A is forced in every world of every Kripke model

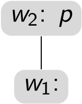
Accordingly, if $A \notin \mathrm{IPL}$, there exists a model $\mathcal{K}$ and a world w in $\mathcal{K}$ such that A is not forced at w.
We call $\mathcal{K}$ a countermodel for A



Semantics

Example

A countermodel for $p \vee \neg p$ is



$$V(w_1) = \emptyset \qquad v(w_2) = \{p\}$$

$w_1 \nVdash p$	since $p \notin V(w_1)$
$w_1 \nVdash \neg p$	since $w_1 \leq w_2$ and $w_2 \Vdash p$ $(p \in V(w_2))$
$w_1 \nVdash p \vee \neg p$	since $w_1 \nVdash p$ and $w_1 \nVdash \neg p$

At w_1, p is not forced.

The world w_1 is followed by a world w_2 and p is forced at w_1, thus $\neg p$ is not forced at w_2. Since forcing must be preserved through $\leq$, $\neg p$ is not forced at w_1.

We conclude that $p \vee \neg p$ is not forced at w_1.

Validity vs. non-validity

Let G be a goal formula

- The validity of G in IPL can be witnessed by a derivation in a sound calculus for IPL
  Hilbert calculus, natural deduction deduction, tableaux/sequent, ...

- The non-validity of G can be witnessed by a countermodel

Typically, the emphasis is on derivations and countermodels are obtained as a result of a failed proof-search for a derivation of G.

For almost all the the known tableaux/sequent calculi for IPL, we can define a proof-search procedure `ProofSearch` such that:

$$\texttt{ProofSearch}(G) = \begin{cases} \text{A derivation of } G & \text{If } G \in \mathrm{IPL} \\ \text{A countermodel for } G & \text{Otherwise} \end{cases}$$

Countermodels

- A countermodel can be understood as a certificate witnessing the non-validity of the goal formula G
- Countermodels can be used for diagnosis, to analyze why some property fails or to fix errors in formal specifications (see Property-Based Testing).
- It is critical that countermodels are minimal so as to convey a plain and concise representation of non-validity.

This issue has been scarcely investigated in the literature. Many proof-seearch procedures have been introduced, but all fail to build small countermodels

L. Pinto and R.Dyckhoff, Loop-free construction of counter-models for intuitionistic propositional logic. Symposia Gaussiana Conf, 1995

G. Corsi and G. Tassi. Intuitionistic logic freed of all metarules. JSL, 2007

M. Ferrari, C. Fiorentini, and G. Fiorino. Contraction-free linear depth sequent calculi for intuitionistic propositional logic with the subformula property and minimal depth counter-models. JAR, 2013.

D. Larchey-Wendling, D. Mry, and D. Galmiche. STRIP: Structural Sharing for Efficient Proof-Search. IJCAR, 2001.

V. Svejdar. On sequent calculi for intuitionistic propositional logic. Comment. Math. Univ. 2006

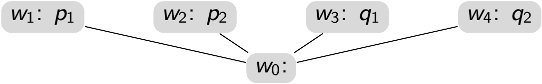
...

Countermodels

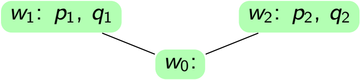
Example

$$G \;=\; (p_1 \to p_2) \vee (p_2 \to p_1) \vee (q_1 \to q_2) \vee (q_2 \to q_1)$$

Countermodel generated by `ProofSearch(G)` [Ferrari et al.,TOCL,2015] generating countermodels of minimal depth



The model has minimal height, but it is not mininal in the number of worlds. A minimum countermodel is:



Note that we cannot shrink the first model to get a minimum one!

Countermodel generation

Main contribution

We present a procedure to generate minimal countermoles:

- given a goal formula G, we try to build a countermodel for G by a model-search procedure guided by semantics.

A naive implementation of the process immediately blows-up; even for small goal formulas, model generation is not terminating.

We need a clever formalization of the problem.

Countermodel generation

- Model formalization

  We follow the approach of R. Goré et al. [IJCAR 2012, 2014]:

  - Worlds of models are represented by sets $\mathcal{W}$ of atomic subformulas H of G, namely:

    $$H ::= p \mid \neg A \mid A \rightarrow B \qquad p\text{: propositional variable}$$

  - We do not considers all possible sets $\mathcal{W}$ of atomic subformulas, but only the sets $\mathcal{W}$ satisfying some closure properties, we call p-worlds (possible worlds)

    For instance:

    $$\mathcal{W}_1 = \{\, p, \neg p \,\} \qquad \mathcal{W}_2 = \{\, p, p \rightarrow q \,\}$$

    $\mathcal{W}_1$ must be discarded since it is inconsistent
    $\mathcal{W}_1$ must be discarded since it is not closed under modus ponens ($q \notin \mathcal{W}_2$)

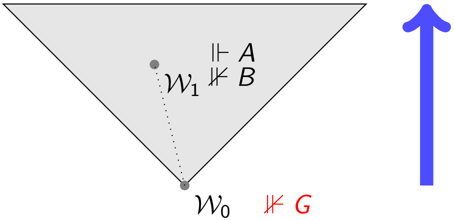
Countermodel generation

- The first selected p-world $\mathcal{W}_0$ is a putative world falsifying G.
- To get a well-defined Kripke model, we have to guarantee that atomic subformulas of G not belonging to $\mathcal{W}_0$ are not valid in $\mathcal{W}_0$, for instance:

$$A \to B \notin \mathcal{W}_0 \quad \Longrightarrow \quad \exists \mathcal{W}_1 (\, \mathcal{W}_0 \subseteq \mathcal{W}_1 \wedge A \in \mathcal{W}_1 \wedge B \notin \mathcal{W}_1 \,)$$

$\mathcal{W}_1$ is needed to witness the non-validity of $A \to B$ in $\mathcal{W}_0$.

This triggers a saturation process which successfully ends when all the needed witnesses have been generated, thus yielding a countermodel for G.



Countermodel generation

- Computation engine

  We formalize the search problem in Answer Set Programming (ASP) [Baral 2010].

  - √ ASP is a form of declarative programming based on the stable model semantics (answer sets),
  - √ ASP enables to solve hard search problems (in NP and in NP^{NP}) in a uniform way

- We define an ASP program Π_G such that an answer set of Π_G corresponds to a countermodel for G.
  If no answer exists, there is no countermodel for G, meaning that G is valid (in IPL).

- To compute answer sets, we exploit the Potassco tool `clingo` [Gebser et al.,2012].

- The minimization of models is delegated to `clingo`; however, it is critical to encode the problem so that even the first computed model is small, otherwise the minimization engine gets stuck.

Countermodel generation

Differently from other declarative formalisms, ASP allows for a quite modular formalization:

$$\Pi_G \;=\; \mathtt{Gen} \;+\; \mathtt{Goal}(G)$$

- `Gen` encodes the genereator and is independent of the goal formula
- $\mathtt{Goal}(G)$ encodes the goal formula

The generator can be easily extended to deal with other intermediate logics where the frame conditions can be expressed in ASP, such as:

- The Gödel-Dummett logic [Dummett,59], characterized by linear frames
- The logic of bound-depth frames
- Here and There logic [Pearce,97], well-known in ASP

Countermodel generation

Frame conditions can be freely composed:

- `lin.lp` encodes the contraint "the model is linear"

```
:-  world(W1), world(W2), W1 <> W2 ,  not le(W1,W2), not le(W2,W1).
```

- `bd2.lp` encodes the contraint "the model has depth at most 2"

```
:-  world(W1), world(W2), world(W3), W1 <> W2, W1 <> W3, W2 <> W3,
    le(W1,W2), le(W2,W3).
```

```
clingo  gen.lp  goal.lp  lin.lp          //  linear countermodels
clingo  gen.lp  goal.lp  bd2.lp     //  detpth <=2 countermodels

clingo  gen.lp  goal.lp  lin.lp  bd2.lp
 //  linear AND  detpth <=2 countermodels
```

This kind of modularity is not possible with derivations!

Countermodel generation

The program is efficiente with formulas containing few propositional variables.
For instance, let us consider the non-valid Nishimura formulas:

$$N_1 = p \qquad N_2 = \neg p$$
$$N_{2n+3} = N_{2n+1} \vee N_{2n+2} \qquad N_{2n+4} = N_{2n+3} \to N_{2n+1}$$

The cuntermodel for N_{17} is computed in few seconds: