Appendix

Lemma 1. \( \sigma' \to \sigma \) implies Lhs(\( \sigma \)) \( \subseteq \) Cl(Lhs(\( \sigma' \))).

Proof. If \( \sigma' \to \sigma \), the assertion follows by the definition of the rules of the calculus and the properties of closures. For instance, let \( \sigma \) be the conclusion of rule \( \lor \) of Fig. 1 and \( \sigma' \) the right premise. Since \( \Sigma_1 \subseteq \Sigma_2 \cup \Theta_2 \), we get Lhs(\( \sigma \)) \( \subseteq \) Lhs(\( \sigma_2 \)), which implies, by \( \text{C} \) Lhs(\( \sigma \)) \( \subseteq \) Cl(Lhs(\( \sigma_2 \))). Having proved the assertion for \( \to \), the generalization to \( \to \) follows by \( \text{C} \).

Lemma 3. Let \( \mathcal{D} \) be an FRJ(G)-derivation and \( \sigma \) a sequent occurring in \( \mathcal{D} \).

(i) If \( \sigma = \Gamma \Rightarrow C \), then \( \phi(\sigma) \models \Gamma \) and \( \phi(\sigma) \not\models C \).

(ii) If \( \sigma = \Sigma; \Theta \Rightarrow C \), let \( \sigma_p \in \mathcal{P}(\mathcal{D}) \) such that \( \sigma \to \sigma_p \) and \( \sigma_p \models \Sigma \cap \text{Sf}^{-}(C) \);
then \( \sigma_p \not\models C \).

Proof. We present the cases not discussed in Sec. 4.

Let \( \mathcal{R} = \square \vDash \) and \( \sigma \) irregular:

\[
\frac{\sigma_1 = \Sigma_1; \theta, \varphi \Rightarrow B \quad \vDash \vDash}{\sigma = \Sigma_1; \theta \Rightarrow A \supset B \quad \vDash} \quad A \in \text{Cl}(\Sigma), \text{ where } \Sigma = \Sigma_1 \cup \Lambda
\]

By hypothesis \( \sigma_p \models \Sigma \cap \text{Sf}^{-}(A \supset B) \), hence \( \sigma_p \models \Sigma_1 \cap \text{Sf}^{-}(B) \) and \( \sigma_p \models \Sigma_A \)
(indeed, \( \text{Sf}(A) \subseteq \text{Sf}^{-}(A \supset B) \)). Since \( \sigma_1 \to \sigma_p \), by (IH1) applied to \( \sigma_1 \) we get \( \sigma_p \not\models B \). Since \( A \in \text{Cl}(\Sigma) \), by \( \text{C} \) we get \( \sigma_p \not\models A \).

We conclude \( \sigma_p \not\models A \supset B \) and (ii) holds.

Let \( \mathcal{R} = \vDash \). Then:

\[
\frac{\sigma_1 = \Gamma \Rightarrow B \quad \vDash}{\sigma = \Gamma \Rightarrow A \supset B \quad \vDash} \quad A \in \text{Cl}(\Gamma)
\]

By (IH1) applied to \( \sigma_1 \), we have \( \phi(\sigma_1) \models \Gamma \) and \( \phi(\sigma_1) \not\models B \). By \( \text{C} \) \( \phi(\sigma_1) \models A \).

Since \( \sigma_1 \to \sigma_p \), we have \( \sigma_p \leq \phi(\sigma_1) \), hence \( \sigma_p \not\models A \supset B \), and this proves (ii).

The case \( \mathcal{R} = \vDash \) is similar to the case \( \mathcal{R} = \vDash \) detailed in Sec. 4. Finally, the case \( \mathcal{R} = \land \) easily follows by (IH1).

To prove Lemma 4, we need the following property of closures:

Lemma 5. Let \( \mathcal{K} \) be a countermodel for \( G \) and \( \alpha \) a world in \( \mathcal{K} \). Then, \( \Lambda_\alpha = \text{Cl}(\Lambda_\alpha) = \text{Cl}(\Lambda_\alpha^*) \).

Proof. By \( \text{C} \) \( \Lambda_\alpha \subseteq \text{Cl}(\Lambda_\alpha) \). By induction on \( |\mathcal{C}| \), one can easily prove that \( C \in \text{Cl}(\Lambda_\alpha) \) implies \( C \in \Lambda_\alpha \), hence \( \Lambda_\alpha = \text{Cl}(\Lambda_\alpha) \). Since \( \Lambda_\alpha^* \subseteq \Lambda_\alpha \), by \( \text{C} \) we get \( \text{Cl}(\Lambda_\alpha^*) \subseteq \text{Cl}(\Lambda_\alpha) \). It remains to prove that \( \Lambda_\alpha \subseteq \text{Cl}(\Lambda_\alpha^*) \).

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Proof. By \( \text{C} \) \( \Lambda_\alpha \subseteq \text{Cl}(\Lambda_\alpha) \). By induction on \( |\mathcal{C}| \), one can easily prove that \( C \in \text{Cl}(\Lambda_\alpha) \) implies \( C \in \Lambda_\alpha \), hence \( \Lambda_\alpha = \text{Cl}(\Lambda_\alpha) \). Since \( \Lambda_\alpha^* \subseteq \Lambda_\alpha \), by \( \text{C} \) we get \( \text{Cl}(\Lambda_\alpha^*) \subseteq \text{Cl}(\Lambda_\alpha) \). It remains to prove that \( \Lambda_\alpha \subseteq \text{Cl}(\Lambda_\alpha^*) \).

Let \( C \in \text{SL}(G) \) such that \( \alpha \vdash C \); by induction on \( |\mathcal{C}| \), we show that \( C \in \text{Cl}(\Lambda_\alpha^*) \). If \( C \in \mathcal{V} \), then \( \alpha \vdash \ast \) \( C \), hence \( C \in \Lambda_\alpha^* \), which implies \( C \in \text{Cl}(\Lambda_\alpha^*) \). Let \( C = A \supset B \). If \( \alpha \not\vdash A \), then \( \alpha \vdash \ast \) \( A \) and, as above, \( C \in \text{Cl}(\Lambda_\alpha^*) \). If \( \alpha \vdash A \), then \( \alpha \vdash B \); by induction hypothesis, \( B \in \text{Cl}(\Lambda_\alpha^*) \), hence \( A \supset B \in \text{Cl}(\Lambda_\alpha^*) \). The cases \( C = A \land B \) and \( C = A \lor B \) easily follow by the induction hypothesis.
The height of a world \( \alpha \) of a model \( K \) is the maximal length of a path from \( \alpha \) to a final world of \( K \).

**Lemma 4.** Let \( K = (P, \leq, \rho, V) \) be a countermodel for \( G \) and \( \alpha \in P \). For every \( C \in \Omega_\alpha \), we can choose \( \Gamma, \Sigma \) and \( \Theta \) such that:

(i) \( \vdash_{\text{FRJ}} (G) \sigma \), where \( \sigma = \Gamma \Rightarrow C \).

(ii) there is \( \beta \in P \) such that \( \alpha \leq \beta \) and \( \Lambda^*_\beta \subseteq \Gamma \).

(iii) \( \vdash_{\text{FRJ}} (G) \sigma \), where \( \sigma = \Sigma ; \Theta \rightarrow C \).

(iv) \( \Sigma \subseteq \Lambda^*_\alpha \subseteq \Sigma \cup \Theta \).

Let \( S_\alpha \) be the set of sequents selected in (i) and (iii) and \( S^*_\alpha \) the union of \( S_\beta \) such that \( \alpha \leq \beta \).

Then, to prove \( \sigma \in S_\alpha \) we only need to use sequents in \( S^*_\alpha \).

**Proof.** Let \( \alpha \in P \) and \( C \in \Omega_\alpha \). We use a main induction (IH1) on \( h(\alpha) \); a secondary induction (IH2) on \( \text{tp}^- (\sigma) \), where \( \text{tp}^- (\sigma) = 1 \) if \( \sigma \) is regular, \( \text{tp}^- (\sigma) = 0 \) otherwise; a third induction (IH3) on \( |C| \).

We proceed by a case analysis on \( C \); we set:

\[
\begin{align*}
T^\text{At} &= \text{Sl}(G) \cap V, \\
T^\varnothing &= \text{Sl}(G) \cap L^\varnothing, \\
T &= T^\text{At} \cup T^\varnothing, \\
\Lambda^*_\alpha &= \Lambda^*_\alpha \cap L^\varnothing.
\end{align*}
\]

In each case, one can easily check that the derivations satisfy the last assertion of the lemma. We also point out that derivations satisfy properties \([\text{PS1}] \ (\text{PS4})\) stated in Sec. 3.

- **Case** \( C \in V^\bot \), proof of (i) and (ii)

Since \( \alpha \nvdash C \), we have \( C \notin \Lambda^*_\alpha \). If \( \Lambda^*_\alpha \) is empty, then \( \Lambda^*_\alpha \subseteq T^\text{At} \setminus \{C\} \). Thus, taking as \( \sigma \) the regular axiom \( T^\text{At} \setminus \{C\} \Rightarrow C \), points (i) and (ii) (where \( \beta = \alpha \)) immediately follow.

Let \( \Lambda^*_\alpha \) be non empty and let \( Y = \{A_1, \ldots, A_n\} \) be the set of formulas \( Y \) such that \( Y \supseteq Z \in \Lambda^*_\alpha \). Note that \( \alpha \nvdash A_j \), for every \( A_j \in Y \). Thus, we can apply (IH2) to claim that, for every \( 1 \leq j \leq n \), there are \( \Sigma_j = \Sigma_j^\text{At} \cup \Sigma_j^\varnothing \) and \( \Theta_j = \Theta_j^\text{At} \cup \Theta_j^\varnothing \) such that:

\[
\begin{align*}
\text{(P3)} & \vdash_{\text{FRJ}} (G) \sigma_j, \text{ where } \sigma_j = \Sigma_j ; \Theta_j \rightarrow A_j. \\
\text{(P4)} & \Sigma_j \subseteq \Lambda^*_\alpha \subseteq \Sigma_j \cup \Theta_j.
\end{align*}
\]

We stress that the use of (IH2) is sound since \( \text{tp}^- (\sigma_j) < \text{tp}^- (\sigma) \). We prove that \( \sigma_1, \ldots, \sigma_n \) satisfy the side conditions of rule \( \forall^\text{At} \). To this aim, we show that, for every \( 1 \leq j \leq n \), the following holds:

\[
\begin{align*}
\text{(a)} & \Sigma_i \subseteq \Sigma_j \cup \Theta_j, \text{ for every } i \neq j. \\
\text{(b)} & Y \supseteq Z \in \Sigma_j^\varnothing \text{ implies } Y \in \text{T}. \\
\text{(c)} & C \notin \Sigma_j^\text{At}.
\end{align*}
\]

Let \( j \in \{1, \ldots, n\} \) and \( i \neq j \). By (P4) we have both \( \Sigma_i \subseteq \Lambda^*_\alpha \) and \( \Lambda^*_\alpha \subseteq \Sigma_j \cup \Theta_j \), and this proves (a). Point (b) immediately follows by (P4) and the definition of
\( T \). Point \((c)\) follows by the fact that \( C \notin A^*_\alpha \) and by \((P4)\) we can apply the rule \( \otimes \) with premises \( \sigma_1, \ldots, \sigma_n \) and build the \( \text{FRJ}(G) \)-derivation:

\[
\begin{array}{c}
  j = 1 \ldots n \\
  \Sigma_j^\text{At}, \sigma_j : \Theta_j^\text{At}, \Theta_j^\supset \rightarrow A_j \ldots \\
  \sigma = \Gamma \Rightarrow C \\
  \end{array}
\]

Thus \((i)\) holds; note that, by the definition of \( T \), the application of \( \otimes \) satisfies \((PS3)\). We show that \( A^*_\alpha \subseteq \Gamma \), and this proves \((ii)\). If, for some \( j \in \{1, \ldots, n\} \), \( A^*_\alpha \subseteq \Sigma_j^\text{At} \cup \Sigma_j^\supset \). Otherwise, by \((P4)\) \( A^*_\alpha \subseteq \bigcap_{1 \leq j \leq n} \Theta_j \). Since \( C \notin A^*_\alpha \), we get \( A^*_\alpha \subseteq (\Theta^\text{At} \setminus \{C\}) \cup \Theta^\supset \). In both cases we conclude \( A^*_\alpha \subseteq \Gamma \).

- Case \( C \in \mathcal{V} \), proof of \((iii)\) and \((iv)\)

Trivial, taking as \( \sigma \) the irregular axiom \(. \, \cdot \, \bigcap \{C\} \Rightarrow C \).

- Case \( C = C_1 \lor C_2 \), proof of \((i)\) and \((iii)\)

Since \( \alpha \not\in C_1 \lor C_1 \), we have \( \alpha \not\in C_1 \) and \( \alpha \not\in C_2 \). By \((I2)\), for \( k \in \{1, 2\} \) there are \( \Sigma_k = \Sigma_k^\text{At} \cup \Sigma_k^\supset \) and \( \Theta_k = \Theta_k^\text{At} \cup \Theta_k^\supset \) such that:

\[
\begin{aligned}
  (Q3) & \vdash \text{FRJ}(G) \Sigma_k, \text{where } \Sigma_k = \Sigma_k; \Theta_k \Rightarrow C_k. \\
  (Q4) & \Sigma_k \subseteq A^*_\alpha \subseteq \Sigma_k \cup \Theta_k. \\
\end{aligned}
\]

If \( A^*_\alpha \) is empty, by \((Q4)\) we have \( \Sigma_k = \Sigma_k^\text{At} \), for \( k \in \{1, 2\} \). Hence, we can build the \( \text{FRJ}(G) \)-derivation

\[
\begin{array}{c}
  \Sigma_1^\text{At}; \Theta_1^\text{At}, \Theta_1^\supset \Rightarrow C_1 \\
  \Sigma_2^\text{At}; \Theta_2^\text{At}, \Theta_2^\supset \Rightarrow C_2 \\
  \sigma = \Sigma_1^\text{At} \cup \Sigma_2^\text{At}, \Theta_1^\text{At} \cap \Theta_2^\text{At} \Rightarrow C_1 \lor C_2 \\
  \end{array}
\]

and this proves \((i)\) By \((Q4)\) we get \( A^*_\alpha \subseteq \Gamma \), which proves \((ii)\). Let \( A^*_\alpha \) be non empty and let \( \mathcal{Y} = \{A_1, \ldots, A_n\} \) be the set of formulas \( Y \) such that either \( Y \supset Z \in A^*_\alpha \) or \( Y = C_1 \) or \( Y = C_2 \). Note that \( \alpha \not\in A_j \), for every \( A_j \in \mathcal{Y} \). Arguing as above, points \((P3)\) and \((P4)\) hold, hence we can build the \( \text{FRJ}(G) \)-derivation

\[
\begin{array}{c}
  j = 1 \ldots n \\
  \Sigma_j^\text{At}, \sigma_j : \Theta_j^\text{At}, \Theta_j^\supset \rightarrow A_j \ldots \\
  \sigma = \Gamma \Rightarrow C_1 \lor C_2 \\
  \end{array}
\]

and this proves \((i)\) Point \((iii)\) (with \( \beta = \alpha \)) can be proved as above, exploiting \((P4)\). We point out that the displayed applications of \( \otimes \) match \((PS4)\).

- Case \( C = C_1 \lor C_2 \), proof of \((iii)\) and \((iv)\)
By (IH3), points (Q3) and (Q4) hold; thus \( \Sigma_1 \subseteq \Sigma_2 \cup \Theta_2 \) and \( \Sigma_2 \subseteq \Sigma_1 \cup \Theta_1 \). This implies that we can apply rule \( \lor \) to \( \sigma_1 \) and \( \sigma_2 \) and get an \( \text{FRJ}(G) \)-derivation of \( \sigma = \Sigma_1, \Sigma_2 : \Theta_1 \cap \Theta_2 \rightarrow C_1 \lor C_2 \), which proves (iii). Point (iv) follows by (Q4).

- Case \( C = C_1 \land C_2 \).

Since \( \alpha \not\vdash C_k \) for \( k \in \{1, 2\} \) such that \( \alpha \not\vdash C_k \). Using (IH3), the assertions easily follow.

- Case \( C = A \cup B \), proof of (i) and (ii).

Since \( \alpha \not\vdash A \cup B \), there is \( \eta \in P \) such that \( \alpha \leq \eta \) and \( \eta \vdash A \) and \( \eta \not\vdash B \). Since \( \eta \not\vdash B \), by induction hypothesis (IH1) if \( \alpha < \eta \) and (IH3) if \( \alpha = \eta \), there is \( \Gamma \) such that:

\[
\begin{align*}
(R1) & \vdash_{\text{FRJ}(G)} \sigma_1, \text{ where } \sigma_1 = \Gamma \Rightarrow B. \\
(R2) & \text{There is } \beta \in P \text{ such that } \alpha \leq \beta \text{ and } A_\beta \subseteq \Gamma.
\end{align*}
\]

We show that \( A \in \text{Cl}(\Gamma) \), so that an application of rule \( \supset \) to \( \sigma_1 \) yields \( \sigma = \Gamma \Rightarrow A \cup B \), and this proves (i). Since \( \eta \leq \beta \), we have \( \beta \vdash A \), namely \( A \in A_\beta \). By Lemma 5, \( A \in \text{Cl}(\Lambda^*_A) \), which implies, by (R2) and (CL4) \( A \in \text{Cl}(\Gamma) \). Point (ii) follows by (R2).

- Case \( C = A \cup B \), proof of (iii) and (iv).

Since \( \alpha \not\vdash A \cup B \), there is \( \eta \in P \) such that \( \alpha \leq \eta \) and \( \eta \vdash A \) and \( \eta \not\vdash B \). Without loss of generality, we assume that, for every \( \delta \in P \) such that \( \alpha \leq \delta < \eta \), we have \( \delta \not\vdash A \). Since \( \alpha \leq \eta \), it holds that \( \alpha \not\vdash B \). By (IH3) there are \( \Sigma_1 \) and \( \Theta_1 \) such that:

\[
\begin{align*}
(S3) & \vdash_{\text{FRJ}(G)} \sigma_1, \text{ where } \sigma_1 = \Sigma_1 : \Theta_1 \rightarrow B. \\
(S4) & \Sigma_1 \subseteq A^*_\alpha \subseteq \Sigma_1 \cup \Theta_1.
\end{align*}
\]

Let \( \eta = \alpha \). Since \( A \in A_\alpha \), by Lemma 3, \( A \in \text{Cl}(\Lambda^*_A) \). As a minimum subset of \( A \subseteq \Lambda^*_A \) such that \( A \in \text{Cl}(\Lambda) \) (namely: \( A' \subseteq \Lambda \) implies \( A \not\in \text{Cl}(A') \)). Note that, by (S4) \( A \subseteq \Sigma_1 \cup \Theta_1 \), hence we can partition \( \Lambda \) as \( A_\Sigma \cup A_\Theta \), as shown below. We can build the following \( \text{FRJ}(G) \)-derivation, where rule \( \supset \) shifts the set \( A_\Theta \) to the left of semicolon:

\[
\begin{align*}
\Sigma_1 \cup A_\Sigma, A_\Theta \vdash B \quad \text{where} \\
A_\Sigma = A \cap \Sigma_1 & \quad A_\Theta = A \cap \Theta_1 \\
\Sigma_2 \Rightarrow A_\Sigma & \quad \Theta_2 = \Theta_1 \setminus A_\Theta \\
\sum \Sigma_2, A_\Sigma, A_\Theta \vdash B & \quad \subseteq \varepsilon
\end{align*}
\]

Since \( A \in \text{Cl}(A) \) and \( A \subseteq \Sigma \), by (CL4) we get \( A \in \text{Cl}(\Sigma) \), hence the application of \( \supset \) is sound and (iii) holds. Since \( \Sigma_1 \subseteq A_\alpha^* \) (see (S4)) and \( A_\Theta \subseteq A \subseteq A_\alpha^* \), we
get $\Sigma_1 \cup \Lambda_\emptyset \subseteq \Lambda^*_\emptyset$, namely $\Sigma \subseteq \Lambda^*_\emptyset$. Moreover, since $\Lambda^*_\emptyset \subseteq \Sigma_1 \cup \Theta_1$ (see [S4]) and $\Sigma_1 \cup \Theta_1 = \Sigma \cup \Theta_2$, we get $\Lambda^*_\emptyset \subseteq \Sigma \cup \Theta_2$, and this concludes the proof of (iv).

We notice that the choice of $\Lambda$ complies with [PS1].

Let $\alpha < \eta$ (hence $h(\eta) < h(\alpha)$). By the choice of $\eta$, we can assume $\alpha \not\models A$.

Since $\eta \not\models B$, by (IH1) there is $\Gamma$ such that:

(T1) $\vdash_{\text{FRJ}(G)} \sigma_1$, where $\sigma_1 = \Gamma \Rightarrow B$.

(T2) There exists $\mu \in P$ s.t. $\eta \leq \mu$ (hence $\alpha < \mu$) and $\Lambda^*_\mu \subseteq \Gamma$.

Since $\eta \models A$ and $\eta \leq \mu$, we get $\mu \models A$, hence $A \in A_\mu$. By Lemma 5 $A \in \text{Cl}(A^*_\mu)$ hence, by [T2] and [CL4], $A \in \text{Cl}(\Gamma)$. Since $\alpha \not\models A$, we have $A \not\in \Lambda_\alpha$ hence, by Lemma 5 $A \not\in \text{Cl}(A^*_\alpha)$. Since $\alpha < \mu$, we have $\Lambda^*_\alpha \subseteq A_\mu$. By Lemma 5 $A_\mu = \text{Cl}(A^*_\mu)$. By [T2] and [CL4] $\text{Cl}(A^*_\mu) \subseteq \text{Cl}(\Gamma)$, hence $\Lambda^*_\alpha \subseteq \text{Cl}(\Gamma)$. Thus $\Lambda^*_\alpha \subseteq \text{Cl}(\Gamma) \cap \overline{\text{Cl}(\Gamma)}$ and $A \not\in \text{Cl}(\Lambda^*_\alpha)$. Let $\Theta$ be a maximum extension of of $\Lambda^*_\alpha$ such that $\Lambda^*_\alpha \subseteq \Theta \subseteq \text{Cl}(\Gamma) \cap \overline{\text{Cl}(\Gamma)}$ and $A \not\in \text{Cl}(\Theta)$ (namely: $\Theta \subseteq \Theta' \subseteq \text{Cl}(\Gamma) \cap \overline{\text{Cl}(\Gamma)}$ implies $A \in \text{Cl}(\Theta)$). We can build the $\text{FRJ}(G)$-derivation:

$$
\frac{
\vdash_{\text{FRJ}(G)} \sigma_1 \quad \theta \subseteq \text{Cl}(\Gamma) \cap \overline{\text{Cl}(\Gamma)}
}{\Gamma \Rightarrow B \quad A \in \text{Cl}(\Gamma) \setminus \text{Cl}(\Theta)}
$$

This proves (iii). The proof of (iv) is immediate. Note that the choice of $\Theta$ matches [PS2].