Appendix

Lemma 1. $\sigma' \mapsto \sigma$ *implies* $\mathrm{Lhs}(\sigma) \subseteq \mathcal{C}l(\mathrm{Lhs}(\sigma'))$.

Proof. If $\sigma' \mapsto_0 \sigma$, the assertion follows by the definition of the rules of the calculus and the properties of closures. For instance, let σ be the conclusion of rule $\vee$ of Fig. 1 and σ' the right premise. Since $\Sigma_1 \subseteq \Sigma_2 \cup \Theta_2$, we get $\mathrm{Lhs}(\sigma) \subseteq \mathrm{Lhs}(\sigma_2)$, which implies, by ($\mathcal{C}l3$), $\mathrm{Lhs}(\sigma) \subseteq \mathcal{C}l(\mathrm{Lhs}(\sigma_2))$. Having proved the assertion for $\mapsto_0$, the generalization to $\mapsto$ follows by ($\mathcal{C}l6$). $\square$

Lemma 3. *Let $\mathcal{D}$ be an* $\mathbf{FRJ}(G)$*-derivation and σ a sequent occurring in $\mathcal{D}$.*

(i) If $\sigma = \Gamma \Rightarrow C$, then $\phi(\sigma) \Vdash \Gamma$ and $\phi(\sigma) \nVdash C$.
(ii) If $\sigma = \Sigma\,;\,\Theta \to C$, let $\sigma_p \in \mathrm{P}(\mathcal{D})$ such that $\sigma \mapsto \sigma_p$ and $\sigma_p \Vdash \Sigma \cap \mathrm{Sf}^-(C)$; then $\sigma_p \nVdash C$.

Proof. We present the cases not discussed in Sec. 4.

Let $\mathcal{R} = \supset_\in$ and σ irregular:

$$\frac{\sigma_1 \;=\; \Sigma_1\,;\,\Theta, \Lambda \to B}{\sigma \;=\; \Sigma_1, \Lambda\,;\,\Theta \to A \supset B}\supset_\in \qquad \begin{array}{l} A \in \mathcal{C}l(\Sigma),\ \text{where } \Sigma \;=\; \Sigma_1 \cup \Lambda \\ \Sigma_A \;=\; \Sigma \cap \mathrm{Sf}(A) \end{array}$$

By hypothesis $\sigma_p \Vdash \Sigma \cap \mathrm{Sf}^-(A \supset B)$, hence $\sigma_p \Vdash \Sigma_1 \cap \mathrm{Sf}^-(B)$ and $\sigma_p \Vdash \Sigma_A$ (indeed, $\mathrm{Sf}(A) \subseteq \mathrm{Sf}^-(A \supset B)$). Since $\sigma_1 \mapsto \sigma_p$, by (IH1) applied to σ_1 we get $\sigma_p \nVdash B$. Since $A \in \mathcal{C}l(\Sigma)$, by ($\mathcal{C}l2$) we get $A \in \mathcal{C}l(\Sigma_A)$ and, by ($\mathcal{C}l1$), $\sigma_p \Vdash A$. We conclude $\sigma_p \nVdash A \supset B$ and (ii) holds.

Let $\mathcal{R} = \supset_\notin$. Then:

$$\frac{\sigma_1 \;=\; \Gamma \Rightarrow B}{\sigma \;=\; \cdot\,;\,\Theta \to A \supset B}\supset_\notin \qquad A \;\in\; \mathcal{C}l(\Gamma)$$

By (IH1) applied to σ_1, we have $\phi(\sigma_1) \Vdash \Gamma$ and $\phi(\sigma_1) \nVdash B$. By ($\mathcal{C}l1$) $\phi(\sigma_1) \Vdash A$. Since $\sigma_1 \mapsto \sigma_p$, we have $\sigma_p \leq \phi(\sigma_1)$, hence $\sigma_p \nVdash A \supset B$, and this proves (ii).

The case $\mathcal{R} = \bowtie^\vee$ is similar to the case $\mathcal{R} = \bowtie^{\mathrm{At}}$ detailed in Sec. 4. Finally, the case $\mathcal{R} = \wedge$ easily follows by (IH1). $\square$

To prove Lemma 4, we need the following property of closures:

Lemma 5. *Let $\mathcal{K}$ be a countermodel for G and α a world in $\mathcal{K}$. Then, $\Lambda_\alpha = \mathcal{C}l(\Lambda_\alpha) = \mathcal{C}l(\Lambda^*_\alpha)$.*

Proof. By ($\mathcal{C}l3$), $\Lambda_\alpha \subseteq \mathcal{C}l(\Lambda_\alpha)$. By induction on $|C|$, one can easily prove that $C \in \mathcal{C}l(\Lambda_\alpha)$ implies $C \in \Lambda_\alpha$, hence $\Lambda_\alpha = \mathcal{C}l(\Lambda_\alpha)$. Since $\Lambda^*_\alpha \subseteq \Lambda_\alpha$, by ($\mathcal{C}l4$) we get $\mathcal{C}l(\Lambda^*_\alpha) \subseteq \mathcal{C}l(\Lambda_\alpha)$. It remains to prove that $\Lambda_\alpha \subseteq \mathcal{C}l(\Lambda^*_\alpha)$. Let $C \in \mathrm{S{\scriptsize L}}(G)$ such that $\alpha \Vdash C$; by induction on $|C|$, we show that $C \in \mathcal{C}l(\Lambda^*_\alpha)$. If $C \in \mathcal{V}$, then $\alpha \Vdash^* C$, hence $C \in \Lambda^*_\alpha$, which implies $C \in \mathcal{C}l(\Lambda^*_\alpha)$. Let $C = A \supset B$. If $\alpha \nVdash A$, then $\alpha \Vdash^* C$ and, as above, $C \in \mathcal{C}l(\Lambda^*_\alpha)$. If $\alpha \Vdash A$, then $\alpha \Vdash B$; by induction hypothesis, $B \in \mathcal{C}l(\Lambda^*_\alpha)$, hence $A \supset B \in \mathcal{C}l(\Lambda^*_\alpha)$. The cases $C = A \wedge B$ and $C = A \vee B$ easily follow by the induction hypothesis. $\square$

The height of a world α of a model $\mathcal{K}$ is the maximal length of a path from α to a final world of $\mathcal{K}$.

Lemma 4. *Let $\mathcal{K} = \langle P, \leq, \rho, V\rangle$ be a countermodel for G and $\alpha \in P$. For every $C \in \Omega_\alpha$, we can choose Γ, Σ and Θ such that:*

(i) $\vdash_{\mathbf{FRJ}(G)} \sigma$, *where* $\sigma = \Gamma \Rightarrow C$.
(ii) there is $\beta \in P$ *such that* $\alpha \leq \beta$ *and* $\Lambda^*_\beta \subseteq \Gamma$.
(iii) $\vdash_{\mathbf{FRJ}(G)} \sigma$, *where* $\sigma = \Sigma\,;\,\Theta \to C$.
(iv) $\Sigma \subseteq \Lambda^*_\alpha \subseteq \Sigma \cup \Theta$.

Let $\mathcal{S}_\alpha$ be the set of sequents selected in (i) and (iii) and $\mathcal{S}^*_\alpha$ the union of $\mathcal{S}_\beta$ such that $\alpha \leq \beta$. Then, to prove $\sigma \in \mathcal{S}_\alpha$ we only need to use sequents in $\mathcal{S}^*_\alpha$.

Proof. Let $\alpha \in P$ and $C \in \Omega_\alpha$. We use a main induction (IH1) on $\mathrm{h}(\alpha)$; a secondary induction (IH2) on $\mathrm{tp}^-(\sigma)$, where $\mathrm{tp}^-(\sigma) = 1$ if σ is regular, $\mathrm{tp}^-(\sigma) = 0$ otherwise; a third induction (IH3) on $|C|$. We proceed by a case analysis on C; we set:

$$\overline{\Gamma}^{\mathrm{At}} = \mathrm{S{\scriptstyle L}}(G) \cap \mathcal{V} \qquad \overline{\Gamma}^{\supset} = \mathrm{S{\scriptstyle L}}(G) \cap \mathcal{L}^{\supset} \qquad \overline{\Gamma} = \overline{\Gamma}^{\mathrm{At}} \cup \overline{\Gamma}^{\supset} \qquad \Lambda^{*\supset}_\alpha = \Lambda^*_\alpha \cap \mathcal{L}^{\supset}$$

In each case, one can easily check that the derivations satisfy the last assertion of the lemma. We also point out that derivations satisfy properties (PS1)–(PS4) stated in Sec. 3.

– Case $C \in \mathcal{V}^\perp$, proof of (i) and (ii).

Since $\alpha \nVdash C$, we have $C \notin \Lambda^*_\alpha$. If $\Lambda^{*\supset}_\alpha$ is empty, then $\Lambda^*_\alpha \subseteq \overline{\Gamma}^{\mathrm{At}} \setminus \{C\}$. Thus, taking as σ the regular axiom $\overline{\Gamma}^{\mathrm{At}} \setminus \{C\} \Rightarrow C$, points (i)-(ii) (where $\beta = \alpha$) immediately follow. Let $\Lambda^{*\supset}_\alpha$ be non empty and let $\Upsilon = \{A_1, \ldots, A_n\}$ be the set of formulas Y such that $Y \supset Z \in \Lambda^{*\supset}_\alpha$. Note that $\alpha \nVdash A_j$, for every $A_j \in \Upsilon$. Thus, we can apply (IH2) to claim that, for every $1 \leq j \leq n$, there are $\Sigma_j = \Sigma^{\mathrm{At}}_j \cup \Sigma^{\supset}_j$ and $\Theta_j = \Theta^{\mathrm{At}}_j \cup \Theta^{\supset}_j$ such that:

(P3) $\vdash_{\mathbf{FRJ}(G)} \sigma_j$, where $\sigma_j = \Sigma_j\,;\,\Theta_j \to A_j$.
(P4) $\Sigma_j \subseteq \Lambda^*_\alpha \subseteq \Sigma_j \cup \Theta_j$.

We stress that the use of (IH2) is sound since $\mathrm{tp}^-(\sigma_j) < \mathrm{tp}^-(\sigma)$. We prove that $\sigma_1, \ldots, \sigma_n$ satisfy the side conditions of rule $\bowtie^{\mathrm{At}}$. To this aim, we show that, for every $1 \leq j \leq n$, the following holds:

(a) $\Sigma_i \subseteq \Sigma_j \cup \Theta_j$, for every $i \neq j$.
(b) $Y \supset Z \in \Sigma^{\supset}_j$ implies $Y \in \Upsilon$.
(c) $C \notin \Sigma^{\mathrm{At}}_j$.

Let $j \in \{1, \ldots, n\}$ and $i \neq j$. By (P4), we have both $\Sigma_i \subseteq \Lambda^*_\alpha$ and $\Lambda^*_\alpha \subseteq \Sigma_j \cup \Theta_j$, and this proves (a). Point (b) immediately follows by (P4) and the definition of

Υ. Point (c) follows by the fact that $C \notin \Lambda^*_\alpha$ and by (P4). By (a)–(c), we can apply the rule $\bowtie^{\mathrm{At}}$ with premises $\sigma_1, \ldots, \sigma_n$ and build the $\mathbf{FRJ}(G)$-derivation:

$$\dfrac{\ldots\ \begin{matrix}\vdots\ (P3)\\ \Sigma^{\mathrm{At}}_j, \Sigma^{\supset}_j\,;\, \Theta^{\mathrm{At}}_j, \Theta^{\supset}_j \to A_j\end{matrix}\ \ldots}{\sigma\ =\ \Gamma \Rightarrow C}\ \bowtie^{\mathrm{At}} \qquad \begin{array}{l} j = 1\ldots n\\ \Sigma^{\mathrm{At}},\ \Sigma^{\supset},\ \Theta^{\mathrm{At}},\ \Theta^{\supset} \text{ as in Fig. 1}\\ \Gamma\ =\ \Sigma^{\mathrm{At}} \cup (\Theta^{\mathrm{At}} \setminus \{C\}) \cup \Sigma^{\supset} \cup \Theta^{\supset}\end{array}$$

Thus (i) holds; note that, by the definition of Υ, the application of $\bowtie^{\mathrm{At}}$ satisfies $(PS3)$. We show that $\Lambda^*_\alpha \subseteq \Gamma$, and this proves (ii). If, for some $j \in \{1, \ldots, n\}$, $\Lambda^*_\alpha \subseteq \Sigma_j$, then $\Lambda^*_\alpha \subseteq \Sigma^{\mathrm{At}} \cup \Sigma^{\supset}$. Otherwise, by (P4), $\Lambda^*_\alpha \subseteq \bigcap_{1 \leq j \leq n} \Theta_j$. Since $C \notin \Lambda^*_\alpha$, we get $\Lambda^*_\alpha \subseteq (\Theta^{\mathrm{At}} \setminus \{C\}) \cup \Theta^{\supset}$. In both cases we conclude $\Lambda^*_\alpha \subseteq \Gamma$.

– Case $C \in \mathcal{V}^{\perp}$, proof of (iii) and (iv).

Trivial, taking as σ the irregular axiom $\cdot\,;\, \overline{\Gamma}^{\mathrm{At}} \setminus \{C\}, \overline{\Gamma}^{\supset} \to C$.

– Case $C = C_1 \vee C_2$, proof of (i) and (ii).

Since $\alpha \nVdash C_1 \vee C_1$, we have $\alpha \nVdash C_1$ and $\alpha \nVdash C_2$. By (IH2), for $k \in \{1, 2\}$ there are $\Sigma_k = \Sigma^{\mathrm{At}}_k \cup \Sigma^{\supset}_k$ and $\Theta_k = \Theta^{\mathrm{At}}_k \cup \Theta^{\supset}_k$ such that:

(Q3) $\vdash_{\mathbf{FRJ}(G)} \sigma_k$, where $\sigma_k\ =\ \Sigma_k\,;\, \Theta_k \to C_k$.
(Q4) $\Sigma_k\ \subseteq\ \Lambda^*_\alpha\ \subseteq\ \Sigma_k \cup \Theta_k$.

If $\Lambda^{*\supset}_\alpha$ is empty, by (Q4) we have $\Sigma_k = \Sigma^{\mathrm{At}}_k$, for $k \in \{1, 2\}$. Hence, we can build the $\mathbf{FRJ}(G)$-derivation

$$\dfrac{\begin{matrix}\vdots\ (Q3)\\ \Sigma^{\mathrm{At}}_1\,;\, \Theta^{\mathrm{At}}_1, \Theta^{\supset}_1 \to C_1\end{matrix} \qquad \begin{matrix}\vdots\ (Q4)\\ \Sigma^{\mathrm{At}}_2\,;\, \Theta^{\mathrm{At}}_2, \Theta^{\supset}_2 \to C_2\end{matrix}}{\sigma\ =\ \Sigma^{\mathrm{At}}_1, \Sigma^{\mathrm{At}}_2,\ \Theta^{\mathrm{At}}_1 \cap \Theta^{\mathrm{At}}_2 \Rightarrow C_1 \vee C_2}\ \bowtie^{\vee} \quad \Gamma\ =\ \Sigma^{\mathrm{At}}_1 \cup \Sigma^{\mathrm{At}}_2 \cup (\Theta^{\mathrm{At}}_1 \cap \Theta^{\mathrm{At}}_2)$$

and this proves (i). By (Q4) we get $\Lambda^*_\alpha\ \subseteq\ \Gamma$, which proves (ii). Let $\Lambda^{*\supset}_\alpha$ be non empty and let $\Upsilon = \{A_1, \ldots, A_n\}$ be the set of formulas Y such that either $Y \supset Z \in \Lambda^{*\supset}_\alpha$ or $Y = C_1$ or $Y = C_2$. Note that $\alpha \nVdash A_j$, for every $A_j \in \Upsilon$. Arguing as above, points (P3) and (P4) hold, hence we can build the $\mathbf{FRJ}(G)$-derivation

$$\dfrac{\ldots\ \begin{matrix}\vdots\ \text{(P3)}\\ \Sigma^{\mathrm{At}}_j, \Sigma^{\supset}_j\,;\, \Theta^{\mathrm{At}}_j, \Theta^{\supset}_j \to A_j\end{matrix}\ \ldots}{\sigma\ =\ \Gamma \Rightarrow C_1 \vee C_2}\ \bowtie^{\vee} \qquad \begin{array}{l} j = 1\ldots n\\ \Sigma^{\mathrm{At}},\ \Sigma^{\supset},\ \Theta^{\mathrm{At}},\ \Theta^{\supset} \text{ as in Fig. 1}\\ \Gamma\ =\ \Sigma^{\mathrm{At}} \cup \Theta^{\mathrm{At}} \cup \Sigma^{\supset} \cup \Theta^{\supset}\end{array}$$

and this proves (i). Point (ii) (with $\beta = \alpha$) can be proved as above, exploiting (P4). We point out that the displayed applications of $\bowtie^{\vee}$ match (PS4).

– Case $C = C_1 \vee C_2$, proof of (iii) and (iv).

By (IH3), points (Q3) and (Q4) hold; thus $\Sigma_1 \subseteq \Sigma_2 \cup \Theta_2$ and $\Sigma_2 \subseteq \Sigma_1 \cup \Theta_1$. This implies that we can apply rule $\vee$ to σ_1 and σ_2 and get an $\mathbf{FRJ}(G)$-derivation of $\sigma = \Sigma_1, \Sigma_2 \,;\, \Theta_1 \cap \Theta_2 \to C_1 \vee C_2$, which proves (iii). Point (iv) follows by (Q4).

– Case $C = C_1 \wedge C_2$.

Since $\alpha \nVdash C_1 \wedge C_2$, there exists $k \in \{1, 2\}$ such that $\alpha \nVdash C_k$. Using (IH3), the assertions easily follow.

– Case $C = A \supset B$, proof of (i) and (ii).

Since $\alpha \nVdash A \supset B$, there is $\eta \in P$ such that $\alpha \leq \eta$ and $\eta \Vdash A$ and $\eta \nVdash B$. Since $\eta \nVdash B$, by induction hypothesis (IH1) if $\alpha < \eta$ and (IH3) if $\alpha = \eta$, there is Γ such that:

(R1) $\vdash_{\mathbf{FRJ}(G)} \sigma_1$, where $\sigma_1 = \Gamma \Rightarrow B$.
(R2) There is $\beta \in P$ such that $\eta \leq \beta$ and $\Lambda^*_\beta \subseteq \Gamma$.

We show that $A \in \mathcal{C}l(\Gamma)$, so that an application of rule $\supset_\in$ to σ_1 yields $\sigma = \Gamma \Rightarrow A \supset B$, and this proves (i). Since $\eta \leq \beta$, we have $\beta \Vdash A$, namely $A \in \Lambda_\beta$. By Lemma 5, $A \in \mathcal{C}l(\Lambda^*_\beta)$, which implies, by (R2) and ($\mathcal{C}l4$), $A \in \mathcal{C}l(\Gamma)$. Point (ii) follows by (R2).

– Case $C = A \supset B$, proof of (iii) and (iv).

Since $\alpha \nVdash A \supset B$, there is $\eta \in P$ such that $\alpha \leq \eta$ and $\eta \Vdash A$ and $\eta \nVdash B$. Without loss of generality, we assume that, for every $\delta \in P$ such that $\alpha \leq \delta < \eta$, we have $\delta \nVdash A$. Since $\alpha \leq \eta$, it holds that $\alpha \nVdash B$. By (IH3) there are Σ_1 and Θ_1 such that:

(S3) $\vdash_{\mathbf{FRJ}(G)} \sigma_1$, where $\sigma_1 = \Sigma_1 \,;\, \Theta_1 \to B$.
(S4) $\Sigma_1 \subseteq \Lambda^*_\alpha \subseteq \Sigma_1 \cup \Theta_1$.

Let $\eta = \alpha$. Since $A \in \Lambda_\alpha$, by Lemma 5 $A \in \mathcal{C}l(\Lambda^*_\alpha)$. Let Λ be a minimum subset of Λ^*_α such that $A \in \mathcal{C}l(\Lambda)$ (namely: $\Lambda' \subsetneq \Lambda$ implies $A \notin \mathcal{C}l(\Lambda')$). Note that, by (S4), $\Lambda \subseteq \Sigma_1 \cup \Theta_1$, hence we can partition Λ as $\Lambda_\Sigma \cup \Lambda_\Theta$, as shown below. We can build the following $\mathbf{FRJ}(G)$-derivation, where rule $\supset_\in$ shifts the set Λ_Θ to the left of semicolon:

$$\dfrac{\begin{array}{c}\vdots\;(S3)\\ \overbrace{\Sigma_2, \Lambda_\Sigma}^{\Sigma_1} \;;\; \overbrace{\Theta_2, \Lambda_\Theta}^{\Theta_1} \to B\end{array}}{\sigma \;=\; \underbrace{\Sigma_2, \Lambda_\Sigma, \Lambda_\Theta}_{\Sigma} \;;\; \Theta_2 \to A \supset B}\supset_\in \qquad \begin{array}{lll} \Lambda = \Lambda_\Sigma \cup \Lambda_\Theta \text{ where} & & \\ \Lambda_\Sigma = \Lambda \cap \Sigma_1 & & \Lambda_\Theta = \Lambda \cap \Theta_1 \\ \Sigma_2 = \Sigma_1 \setminus \Lambda_\Sigma & & \Theta_2 = \Theta_1 \setminus \Lambda_\Theta \\ \Sigma = \Sigma_2 \cup \Lambda_\Sigma \cup \Lambda_\Theta = \Sigma_1 \cup \Lambda_\Theta & & \end{array}$$

Since $A \in \mathcal{C}l(\Lambda)$ and $\Lambda \subseteq \Sigma$, by ($\mathcal{C}l4$) we get $A \in \mathcal{C}l(\Sigma)$, hence the application of $\supset_\in$ is sound and (iii) holds. Since $\Sigma_1 \subseteq \Lambda^*_\alpha$ (see (S4)) and $\Lambda_\Theta \subseteq \Lambda \subseteq \Lambda^*_\alpha$, we get $\Sigma_1 \cup \Lambda_\Theta \subseteq \Lambda^*_\alpha$, namely $\Sigma \subseteq \Lambda^*_\alpha$. Moreover, since $\Lambda^*_\alpha \subseteq \Sigma_1 \cup \Theta_1$ (see (S4)) and $\Sigma_1 \cup \Theta_1 = \Sigma \cup \Theta_2$, we get $\Lambda^*_\alpha \subseteq \Sigma \cup \Theta_2$, and this concludes the proof of (iv). We notice that the choice of Λ complies with (PS1).

Let $\alpha < \eta$ (hence $\mathrm{h}(\eta) < \mathrm{h}(\alpha)$). By the choice of η, we can assume $\alpha \nVdash A$. Since $\eta \nVdash B$, by (IH1) there is Γ such that:

(T1) $\vdash_{\mathbf{FRJ}(G)} \sigma_1$, where $\sigma_1 = \Gamma \Rightarrow B$.
(T2) There exists $\mu \in P$ s.t. $\eta \leq \mu$ (hence $\alpha < \mu$) and $\Lambda^*_\mu \subseteq \Gamma$.

Since $\eta \Vdash A$ and $\eta \leq \mu$, we get $\mu \Vdash A$, hence $A \in \Lambda_\mu$. By Lemma 5 $A \in \mathcal{C}l(\Lambda^*_\mu)$ hence, by (T2) and ($\mathcal{C}l4$), $A \in \mathcal{C}l(\Gamma)$. Since $\alpha \nVdash A$, we have $A \notin \Lambda_\alpha$ hence, by Lemma 5, $A \notin \mathcal{C}l(\Lambda^*_\alpha)$. Since $\alpha < \mu$, we have $\Lambda^*_\alpha \subseteq \Lambda_\mu$. By Lemma 5 $\Lambda_\mu = \mathcal{C}l(\Lambda^*_\mu)$. By (T2) and ($\mathcal{C}l4$), $\mathcal{C}l(\Lambda^*_\mu) \subseteq \mathcal{C}l(\Gamma)$, hence $\Lambda^*_\alpha \subseteq \mathcal{C}l(\Gamma)$. Thus $\Lambda^*_\alpha \subseteq \mathcal{C}l(\Gamma) \cap \overline{\Gamma}$ and $A \notin \mathcal{C}l(\Lambda^*_\alpha)$. Let Θ be a maximum extension of of Λ^*_α such that $\Lambda^*_\alpha \subseteq \Theta \subseteq \mathcal{C}l(\Gamma) \cap \overline{\Gamma}$ and $A \notin \mathcal{C}l(\Theta)$ (namely: $\Theta \subsetneq \Theta' \subseteq \mathcal{C}l(\Gamma) \cap \overline{\Gamma}$ implies $A \in \mathcal{C}l(\Theta)$). We can build the $\mathbf{FRJ}(G)$-derivation:

$$\dfrac{\begin{array}{c}\vdots \; (T1) \\ \Gamma \Rightarrow B\end{array}}{\sigma \;=\; \cdot\,;\, \Theta \to A \supset B} \supset_{\notin} \qquad \begin{array}{l}\Theta \;\subseteq\; \mathcal{C}l(\Gamma) \,\cap\, \overline{\Gamma} \\ A \,\in\, \mathcal{C}l(\Gamma) \setminus \mathcal{C}l(\Theta)\end{array}$$

This proves (iii). The proof of (iv) is immediate. Note that the choice of Θ matches (PS2) $\square$