

Heuristic Algorithms

Master's Degree in Computer Science/Mathematics

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Web page: **<https://homes.di.unimi.it/cordone/courses/2026-ae/2026-ae.html>**

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Extending the local search without worsening

Instead of repeating the local search, extend it beyond the local optimum

To avoid worsening solutions, the selection step must be modified

$$\tilde{x} := \arg \min_{x' \in N(x)} f(x')$$

and two main strategies allow to do that

- the *Variable Neighbourhood Descent (VND)* changes the neighbourhood N
 - it guarantees an evolution with no cycles (*the objective improves*)
 - it terminates when all neighbourhoods have been exploited
- the *Dynamic Local Search (DLS)* changes the objective function f (\tilde{x} is better than x for the new objective, possibly worse for the old)
 - it can be trapped in loops (*the new objective changes over time*)
 - it can proceed indefinitely

Variable Neighbourhood Descent (VND)

The *Variable Neighbourhood Descent* of Hansen and Mladenović (1997) exploits the fact that a solution is locally optimal for a specific neighbourhood

- a local optimum can be improved using a different neighbourhood

Given a family of neighbourhoods $N_1, \dots, N_{s_{\text{tot}}}$

- 1 set $s := 1$
- 2 apply a *steepest descent* exchange heuristic and find a local optimum \bar{x} with respect to N_s
- 3 flag all neighbourhoods for which \bar{x} is locally optimal and update s
- 4 if \bar{x} is a local optimum for all N_s , terminate; otherwise, go back to point 2

```
Algorithm VariableNeighbourhoodDescent( $I, x^{(0)}$ )
  flags := false  $\forall k$ ;
   $\bar{x} := x^{(0)}$ ;  $x^* := x^{(0)}$ ;  $s := 1$ ;
  While  $\exists s : \text{flag}_s = \text{false}$  do
     $\bar{x} := \text{SteepestDescent}(\bar{x}, s)$ ; { possibly truncated }
    flags := true;
    If  $f(\bar{x}) < f(x^*)$ 
      then  $x^* := \bar{x}$ ; flags' := false  $\forall s' \neq s$ ;
     $s := \text{Update}(s, \text{flag})$ ;
  EndWhile;
  Return  $(x^*, f(x^*))$ ;
```

Using many neighbourhoods means that some might be

- rather large
- slow to explore

In order to increase the efficiency of the method one can

- adopt a **first-best strategy in the larger neighbourhoods**
- **terminate the Steepest Descent before reaching a local optimum**
(possibly even after a single step)

Larger neighbourhoods mainly aim to **move out of the basins of attraction of smaller neighbourhoods**

There is of course a strict relation between VND and VNS
(in fact, they were proposed in the same paper)

The fundamental differences are that in the basic VND

- at each step the current solution is the best known one
- the neighbourhoods are explored,
instead of being used to extract random solutions

They are never huge

- the neighbourhoods do not necessarily form a hierarchy

The update of s is not always an increment

- when a local optimum for each N_s has been reached, terminate

VND is deterministic and would not find anything else

Neighbourhood update strategies for the VND

There are two main classes of *VND* methods

- methods with **heterogeneous neighbourhoods**
 - exploit the potential of topologically different neighbourhoods (e.g., exchange vertices instead of edges)

Consequently, s periodically scans the values from 1 to s_{tot}
(possibly randomly permuting the sequence at each repetition)

- methods with **hierarchical neighbourhoods** ($N_1 \subset \dots \subset N_{s_{\text{tot}}}$)
 - fully exploit the small and fast neighbourhoods
 - resort to the large and slow ones only to get out of local optima
(usually terminating *SteepestDescent* prematurely)

Consequently, the update of s works as in the *VNS*

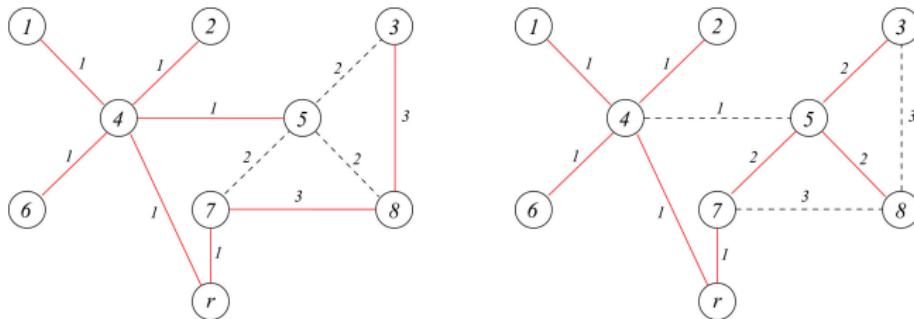
- when no improvements can be found in N_s , increase s
- when improvements can be found in N_s , decrease s back to 1

Terminate when the current solution is a local optimum for all N_s

- in the heterogeneous case, terminate when all fail
- in the hierarchical case, terminate when the largest fails

Example: the CMSTP

This instance of *CMSTP* has $n = 9$ vertices, uniform weights ($w_v = 1$), capacity $W = 5$ and the reported costs (the missing edges have $c_e \gg 3$)



The first solution is locally optimal for N_{S_1} (single-edge swaps):

- swapping any edge in the left branch increases the total cost
- swapping any edge in the right branch
 - with an edge linking to the left one makes the solution unfeasible
 - with an edge linking to the root increases the total cost

Neighbourhood N_{T_1} (single-vertex transfers) has an improving solution, obtained moving vertex 5 from the left branch to the right one

Dynamic Local Search (DLS)

The *Dynamic Local Search* is also known as *Guided Local Search*

Its approach is complementary to *VND*

- it keeps the starting neighbourhood
- it modifies the objective function

It is often used when the objective is useless because it has wide *plateaus*

The basic idea is to

- define a **penalty function** $w : X \rightarrow \mathbb{N}$
- build an **auxiliary function** $\tilde{f}(f(x), w(x))$
which combines the objective function f with the penalty w
- apply a **steepest descent** exchange heuristic to optimise \tilde{f}
- at each iteration **update the penalty w** based on the results

The penalty is adaptive in order to move away from recent local optima
but this introduces the risk of cycling

General scheme of the *DLS*

```
Algorithm DynamicLocalSearch( $I, x^{(0)}$ )
   $w := \text{StartingPenalty}(I);$ 
   $\bar{x} := x^{(0)}; x^* := x^{(0)};$ 
  While  $\text{Stop}() = \text{false}$  do
     $(\bar{x}, x_f) := \text{SteepestDescent}(\bar{x}, f, w); \{ \text{possibly truncated} \}$ 
    If  $f(x_f) < f(x^*)$  then  $x^* := x_f;$ 
     $w := \text{UpdatePenalty}(w, \bar{x}, x^*);$ 
  EndWhile;
  Return  $(x^*, f(x^*));$ 
```

Notice that the *steepest descent* heuristic

- optimises a combination \tilde{f} of f and w
- returns two solutions:
 - ➊ a final solution \bar{x} , locally optimal with respect to \tilde{f} , to update w
 - ➋ a solution x_f , that is the best it has found with respect to f

Variants

The penalty can be applied (for example)

- **additively** to the elements of the solution:

$$\tilde{f}(x) = f(x) + \sum_{i \in x} w_i$$

- **multiplicatively** to components of the objective $f(x) = \sum_j \phi_j(x)$:

$$\tilde{f}(x) = \sum_j w_j \phi_j(x)$$

The penalty can be updated

- at each single neighbourhood exploration
- when a local optimum for \tilde{f} is reached
- when the best known solution x^* is unchanged for a long time

The penalty can be modified with

- **random updates**: “noisy” perturbation of the costs
- **memory-based updates**, favouring the most frequent elements (intensification) or the less frequent ones (diversification)

Example: DLS for the MCP

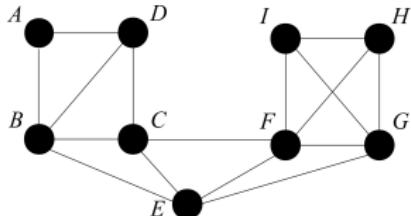
Given a undirected graph, find a maximum cardinality clique

- the exchange heuristic is a *VND* using the neighbourhoods
 - N_{A_1} (vertex addition): the solution always improves, but the neighbourhood is very small and often empty
 - N_{S_1} (exchange of an internal vertex with an external one): the neighbourhood is larger, but forms a *plateau* (uniform objective)
- the objective provides no useful direction in either neighbourhood
- associate to each vertex i a penalty w_i initially equal to zero
- the exchange heuristic minimises the total penalty (within the neighbourhood!)
- update the penalty
 - when the exploration of N_{S_1} terminates: the penalty of the current clique vertices increases by 1
 - after a given number of explorations: all the nonzero penalties decrease by 1

The rationale of the method consists in aiming to

- expel the internal vertices (diversification)
- in particular, the oldest internal vertices (memory)

Example: DLS for the MCP



Start from $x^{(0)} = \{B, C, D\}$, with $w = [0 1 1 1 0 0 0 0 0]$

- ① $w(\{B, C, E\}) = w(\{A, B, D\}) = 2$, but $\{A, B, D\}$ wins lexicographically:
 $x^{(1)} = \{A, B, D\}$ with $w = [1 2 1 2 0 0 0 0 0]$
- ② $x^{(2)} = \{B, C, D\}$ with $w = [1 3 2 3 0 0 0 0 0]$ is the only neighbour
- ③ $w(\{B, C, E\}) = 5 < 7 = w(\{A, B, D\})$:
 $x^{(3)} = \{B, C, E\}$ with $w = [1 4 3 3 1 0 0 0 0]$
- ④ $w(\{C, E, F\}) = 4 < 10 = w(\{B, C, D\})$:
 $x^{(4)} = \{C, E, F\}$ with $w = [1 4 4 3 2 1 0 0 0]$
- ⑤ $w(\{E, F, G\}) = 3 < 11 = w(\{B, C, E\})$:
 $x^{(5)} = \{E, F, G\}$ with $w = [1 4 4 3 3 2 1 0 0]$
- ⑥ $w(\{F, G, H\}) = w(\{F, G, I\}) = 3 < 9 = w(\{C, E, F\})$:
 $x^{(6)} = \{F, G, H\}$ with $w = [1 4 4 3 3 3 2 1 0]$

Now the neighbourhood N_{A_1} is not empty: $x^{(7)} = \{F, G, H, I\}$

Example: DLS for the MAX-SAT

Given m logical disjunctions depending on n logical variables, find a truth assignment satisfying the maximum number of clauses

- neighbourhood N_{F_1} (1-flip) is generated complementing a variable
- associate to each logical clause a penalty w_j initially equal to 1
(each component is a satisfied formula)
- the exchange heuristic maximizes the weight of satisfied clauses thus modifying their number with the multiplicative penalty
- the penalty is updated
 - ① increasing the weight of unsatisfied clauses to favour them

$$w_j := \alpha_{us} w_j \text{ for each } j \in U(x) \quad (\text{with } \alpha_{us} > 1)$$

when a local optimum is reached

- ② reducing the penalty towards 1

$$w_j := (1 - \rho) w_j + \rho \cdot 1 \text{ for each } j \in C \quad (\text{with } \rho \in (0, 1))$$

with a certain probability or after a certain number of updates

Example: DLS for the MAX-SAT

The rationale of the method consists in aiming to

- satisfy the currently unsatisfied clauses (diversification)
- in particular, those which have been unsatisfied for longer time and more recently (memory)

The parameters tune intensification and diversification

- small values of α_{us} and ρ preserve the current penalty (intensification)
- large values of α_{us} push away from the current solution (diversification)
- large values of ρ push towards the local optimum of the current attraction basin (a different kind of intensification)