Heuristic Algorithms

Master's Degree in Computer Science/Mathematics

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What to do when the axioms are violated

If a search space violates the desired axioms, one can try and change it

For the *TSP*, an alternative $\mathcal{F}_A$ includes all paths starting from node 1
Let N_x be the set of nodes visited from x: the acceptable extensions are
all arcs going out of the last node of path x and not closing a subtour

$$\Delta_A^+(x) = \{(h,k) \in A : h = \text{Last}(x), k \notin N_x \text{ or } k = 1 \text{ and } N_x = N\}$$

Unfortunately, the axioms are still not all satisfied

- the trivial axiom always holds
- the accessibility axiom holds: removing the last arc yields a path starting from node 1
- the hereditarity axiom does not hold: not all subsets are paths
- the exchange axiom does not hold (not a greedoid, therefore)

Therefore, it is not even a greedoid

But the algorithm can still be a reasonable heuristic

The *Nearest Neighbour* heuristic for the *TSP*

The *Nearest Neighbour* (*NN*) heuristic adopts the alternative search space keeping the objective function as the selection criterium

- Start with an empty set of arcs: $x^{(0)} = \emptyset$
  that represents a degenerate path going out of node 1
  (*the optimal solution certainly visits node* 1)
- Find the arc of minimum cost going out of the last node of x

$$(i,j) = \arg \min_{(h,k)\in\Delta_A^+(x)} c_{hk}$$

(*the objective function is additive*)

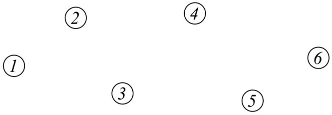
- If $j \neq 1$, go back to point 2; otherwise, terminate
  ($\Delta_A^+(x)$ *allows the return to node* 1 *only at the last step*)

The algorithm is very intuitive and its complexity is $\Theta\left(n^2\right)$

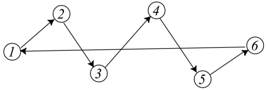
It is not exact, but $\log n$-approximated (*under the triangle inequality*)

The *Nearest Neighbour* heuristic: example

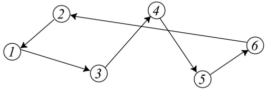
Consider a complete graph (the arcs are not reported for clarity)



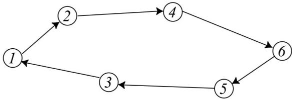
Starting from node 1



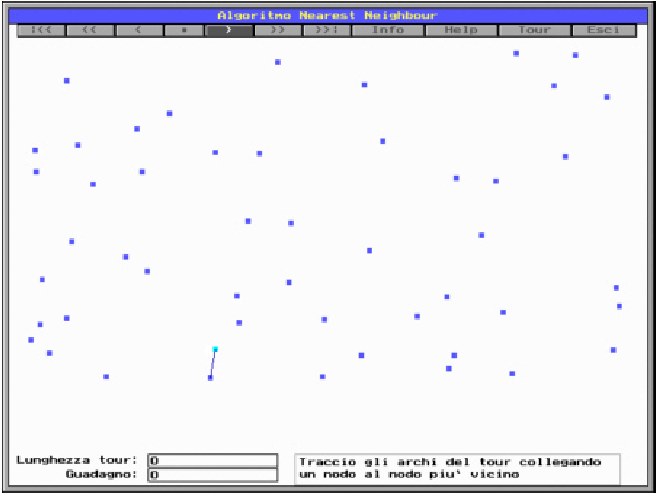
Starting from node 2



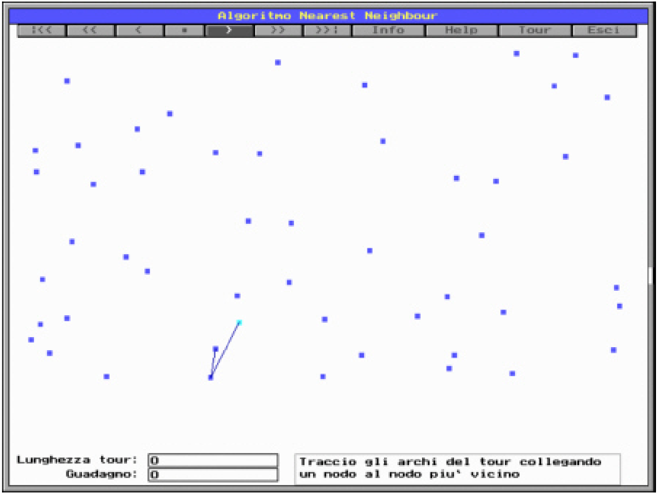
The optimal solution cannot be found starting from any node



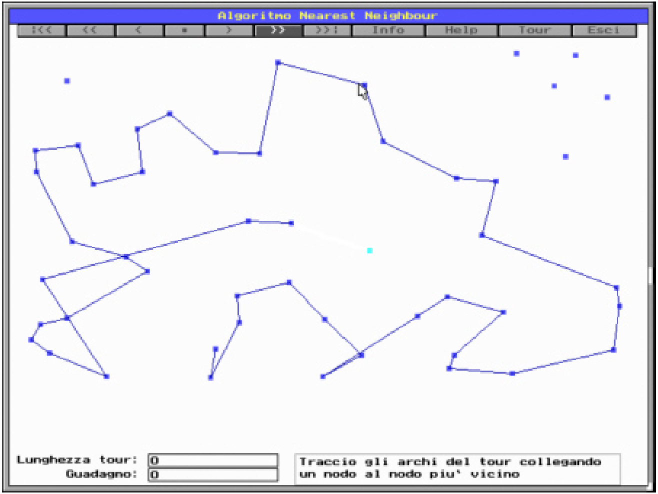
A larger example



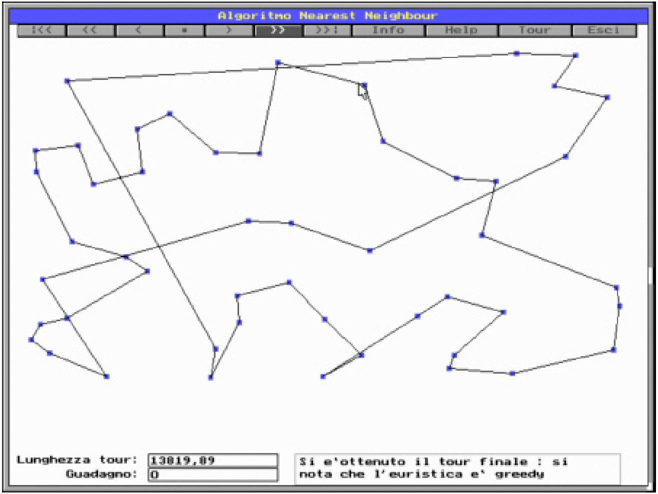
A larger example



A larger example



A larger example



Heuristic constructive algorithms: the *KP*

If the problem does not admit a search space with suitable properties, one must keep into account the constraints of the problem adopting

1. not only a good definition of $\mathcal{F}_A$
2. but also a sophisticated definition of the selection criterium $\varphi_A(i, x)$

This allows effective results, even if not provably optimal

In the *KP*, the drawback derives from the volume of the objects: promising objects have a large value, but also a small volume

- define the selection criterium as the unitary value $\varphi_A(i, x) = \frac{\phi_i}{v_i}$

The resulting algorithm

- can perform very badly
- with a small modification is 2-approximated

Example: the *KP*

B	a	b	c	d	e	f
ϕ	7	2	4	5	4	1
v	5	3	2	3	1	1
ϕ/v	1.40	0.67	2.00	1.67	4	1

$$V = 8$$

The algorithm performs the following steps:

1. $x := \emptyset$;
2. select $i := e$ and update $x := \{e\}$;
3. select $i := c$ and update $x := \{c, e\}$;
4. select $i := d$ and update $x := \{c, d, e\}$;
5. select $i := f$ and update $x := \{c, d, e, f\}$; (*object a does not fit*)
6. since $\Delta_A^+(x) = \emptyset$, terminate

The value of the solution found is 14,
the optimal solution is $x^* = \{a, c, e\}$ and its value is 15

Example: the *KP*

There are critical cases

B	a	b
ϕ	10	90
v	1	10
ϕ/v	10	9

$$V = 10$$

The algorithm performs the following steps:

1. $x := \emptyset$;
2. select $i := a$ and update $x := \{a\}$;
3. since $\Delta_A^+(x) = \emptyset$, terminate

The value of the solution found is 10, the optimum is 90:
there are istances with unlimitedly worse gaps

The reason of the mistake is that

- the first discarded object
- has a large volume, but also a large value

Example: 2-approximated algorithm for the *KP*

1. Start with an empty subset: $x^{(0)} = \emptyset$
2. Find the object $i^{(t)}$ of maximum unitary value in $B \setminus x^{(t-1)}$:

$$i^{(t)} := \arg \max_{i \in B \setminus x^{(t-1)}} \frac{\phi_{i^{(t)}}}{v_{i^{(t)}}}$$

3. If it respects the capacity, add $i^{(t)}$ to $x^{(t-1)}$: $x^{(t)} := x^{(t-1)} \cup \{i^{(t)}\}$ and go back to point 2
4. Build a solution with the first rejected object: $x' = \{i^{(t)}\}$
5. Return the better solution between x and x': $f_A = \max [f(x), f(x')]$

It is easy to prove that

- the sum of the two solution values overestimates the optimum

$$f(x) + f(x') = \sum_{\tau=1}^{t} \phi_{i^{(\tau)}} \geq f^*$$

- the best of the two solution values is at least half their sum

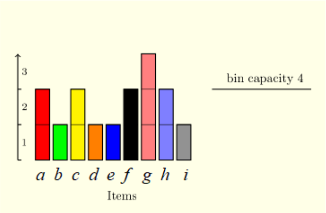
$$f_A = \max [f(x), f(x')] \geq \frac{f(x) + f(x')}{2} \geq \frac{1}{2} f^*$$

Bin Packing Problem

The *BPP* requires to divide a set O of voluminous objects into the minimum number of containers of given capacity drawn from a set C

$B = O \times C$ includes the object-container assignments (i, j)

- with exactly one container for each object
- with the total volume in each container not exceeding the capacity



Let us define the search space $\mathcal{F}_A$ as the set of all partial solutions

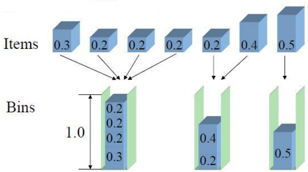
The objective function is a bad selection criterium, because it is flat

All the augmented subsets have the same value or increase it by 1

First-Fit heuristic

Consider the object-container pairs lexicographically

- Start with an empty subset: $x^{(0)} = \emptyset$
- Select pair (i, j) according to the following criterium:
  - i is the first (minimum index) unassigned object
  - j is the first container with enough residual capacity for i (a used container, if possible; an unused one otherwise)
- Add the new assignment to the solution: $x^{(t)} := x^{(t-1)} \cup \{(i, j)\}$

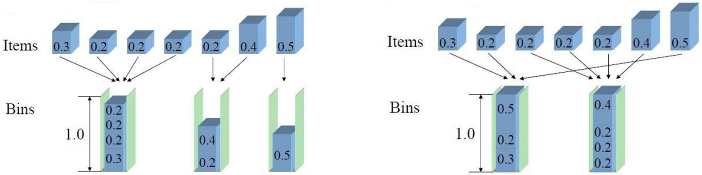


Notice that the choice of (i, j)

- does not minimise $f(x \cup \{(i, j)\})$ *(another i could be better)*
- is split into two phases *(first i, then j)*

Properties of the First-Fit heuristic

The solution is not optimal



but it is approximated:

- at least $f^* \geq \sum_{i \in O} v_i / V$ containers are necessary
- the occupied volume is $> V/2$ for all used containers, possibly except for the last one (*if a second half-empty container existed, its objects would have been assigned to the first*)
- the total volume exceeds that of the $f_A - 1$ "saturated" containers

$$\sum_{i \in O} v_i > (f_A - 1)\,\frac{V}{2}$$

which implies $(f_A - 1) < \frac{2}{V} \sum_{i \in O} v_i \leq 2f^* \Rightarrow f_A \leq 2f^*$

(*the analysis can be improved to* 1.7)

Decreasing First-Fit heuristic

The approximation ratio $\alpha = 2$ holds for any permutation of the objects

Intuition would suggest to select first the smallest objects,
in order to keep the objective $f(x \cup \{i\})$ as small as possible,
but this neglects that all objects must be assigned

By contrast, it is better to select the largest object first because

- each object in a container has a volume strictly larger than the residual capacity of all the previous containers
  (*otherwise, it would have been assigned to one of them*)
- keeping the smallest objects in the end guarantees that many containers have a small residual capacity

This algorithm has a better approximation ratio: $f_A \leq \frac{11}{9} f^* + 1$

Pure and adaptive constructive algorithms

A constructive algorithm A is

- pure if the selection criterium φ_A depends only on the new element i
- adaptive if φ_A depends both on i and on the current solution x

Many criteria $\varphi_A(i,x)$ admit equivalent forms depending only on i

- in the *TSP*, $\varphi_A((i,j),x) = f(x \cup \{(i,j)\})$ is equivalent to c_{ij}
- in the *KP*, $\varphi_A(i,x) = f(x \cup \{i\})$ is equivalent to ϕ_i

So far, we have seen only pure constructive algorithms

An additive selection criterium yields a pure constructive algorithm

Set Covering

Given a binary matrix and a cost vector associated to the columns, find a minimum cost subset of columns covering all the rows

The objective is additive, but the solutions are not maximal subsets (*actually, the smaller feasible subsets are better*)

An adaptive selection criterium $\varphi_A(i, x)$ is necessary: a pure one $(\varphi_A(i))$ could repeatedly choose columns covering the same rows

The more promising ideas are to consider

- the objective function: select columns of low cost
- the constraints: select columns covering many rows
- the current subset x: select columns covering new rows

In summary

- include in $\Delta_A^+(x)$ only columns covering additional rows not in x
- apply the adaptive selection criterium $\varphi_A(i, x) = \dfrac{c_i}{a_i(x)}$

  where $a_i(x)$ is the number of rows covered by i, but not by x

Set Covering: a positive example

c	3	5	6	2	1	7	1	8

	1	0	1	0	0	0	0	1
	0	1	0	0	0	1	0	0
A	1	0	0	1	0	0	0	0
	1	0	1	0	0	1	0	0
	0	0	0	0	1	0	1	1
	1	0	0	1	0	1	0	0

The algorithm performs the following steps:

1. $x := \emptyset$;
2. select $i := 1$ ($\varphi_A(i, x) = 3/4$) and update $x := \{1\}$ and $\Delta_A^+(x) = \{2, 5, 6, 7, 8\}$;
3. select $i := 5$ ($\varphi_A(i, x) = 1$) and update $x := \{1, 5\}$ and $\Delta_A^+(x) = \{2, 6\}$;
4. select $i := 2$ ($\varphi_A(i, x) = 5$) and update $x := \{1, 2, 5\}$;
5. now all the rows are covered and $\Delta_A^+(x) = \emptyset$, therefore terminate

The value of the solution found is $3 + 5 + 1 = 9$ and is the optimum

Set Covering: a negative example

But the algorithm can also fail

c	25	6	8	24	12

	1	1	0	0	0
	1	1	0	0	0
A	1	1	1	0	0
	1	0	1	1	0
	1	0	0	1	0
	1	0	0	0	1

The algorithm performs the following steps:

1. $x := \emptyset$;
2. since $c/a_i(x) = [\,4.1\overline{6} \quad 2 \quad 4 \quad 12 \quad 12\,]$, select $i := 2$;
3. since $c/a_i(x) = [\,8.\overline{3} \quad - \quad 8 \quad 12 \quad 12\,]$, select $i := 3$;
4. since $c/a_i(x) = [\,12.5 \quad - \quad - \quad 24 \quad 12\,]$, select $i := 5$;
5. since $c/a_i(x) = [\,25 \quad - \quad - \quad 24 \quad -\,]$, select $i := 4$;
6. all the rows are covered, therefore $\Delta_A^+(x) = \emptyset$ and terminate

The solution returned is $x = \{2, 3, 4, 5\}$ and its value is 50, whereas the optimal solution $x^* = \{1\}$ has value $f^* = 25$

Approximability of the *SCP*

This algorithm has a nonconstant (logarithmic) approximation ratio

- at each step t, each column i is evaluated with criterium

$$\varphi_A\left(i, x^{(t-1)}\right) = \frac{c_i}{a_i\left(x^{(t-1)}\right)}$$

- each row j is covered by a certain column (i_j) at a certain step (t_j)
- start assigning weight $\theta_j = 0$ to each row j
- when each row j is covered (step t_j), set its weight to

$$\theta_j = \frac{c_{i_j}}{a_{i_j}\left(x^{(t_j-1)}\right)}$$

  so that the total weight of the rows increases by c_{i_j} at step t_j; correspondingly, x includes column i_j and its cost increases by c_{i_j}
- the total cost of x is always equal to the total weight of the rows

$$f_A(x) = \sum_{i \in x} c_i = \sum_{j \in R} \theta_j$$

Approximability of the *SCP*

- at step t, there are $|R^{(t)}|$ uncovered rows
- the columns of the optimal solution could cover them all with cost f^*
  $\Rightarrow$ at least one of such columns has unitary cost $\leq f^*/|R^{(t)}|$
- the column i selected has minimum unitary cost $\varphi_A\left(i, x^{(t-1)}\right)$, therefore $\leq f^*/|R^{(t)}|$ and the covered rows increase their weight by

$$\theta_j = \varphi_A\left(i, x^{(t-1)}\right) \leq \frac{f^*}{|R^{(t_j)}|} \Rightarrow \sum_{j \in R} \theta_j \leq \sum_{j \in R} \frac{f^*}{|R^{(t_j)}|}$$

  The cost to cover each row j is not larger than the optimum divided by the number of rows uncovered at the step in which j gets covered
- the integer numbers $|R^{(t)}|$ are all different
- the sum of all values $\sum_{r=1}^{|R|} \frac{1}{r}$ overestimates $\sum_{j \in R} \frac{1}{|R^{(t_j)}|}$
- The approximation ratio is limited by a logarithmic guarantee

$$f_A = \sum_{j \in R} \theta_j \leq \sum_{j \in R} \frac{f^*}{|R^{(t_j)}|} \leq \sum_{r=|R|}^{1} \frac{f^*}{r} \leq (\ln |R| + 1) \; f^*$$

Application to the negative example

c	25	6	8	24	12

	1	1	0	0	0
	1	1	0	0	0
A	1	1	1	0	0
	1	0	1	1	0
	1	0	0	1	0
	1	0	0	0	1

1. since $\varphi_A(i,x) = \left[\, 4.1\bar{6} \quad 2 \quad 4 \quad 12 \quad 12 \,\right]$, select $i := 2$
   and set $\theta_1 = \theta_2 = \theta_3 = 2$, that is $\leq f^*/|R^{(0)}| = 25/6 = 4.1\bar{6}$;
   now weight $\theta_1 \leq 25/6$, and even more so $\theta_2 \leq 25/5$ and $\theta_3 \leq 25/4$
2. since $\varphi_A(i,x) = \left[\, 8.\bar{3} \quad - \quad 8 \quad 12 \quad 12 \,\right]$, select $i := 3$
   and set $\theta_4 = 8$, that is $\leq f^*/|R^{(1)}| = 8.\bar{3}$
3. since $\varphi_A(i,x) = [\, 12.5 \quad - \quad - \quad 24 \quad 12 \,]$, select $i := 5$
   and set $\theta_6 = 12$, that is $\leq f^*/|R^{(2)}| = 12.5$
4. since $\varphi_A(i,x) = [\, 25 \quad - \quad - \quad 24 \quad - \,]$, select $i := 4$
   and set $\theta_5 = 24$, that is $\leq f^*/|R^{(3)}| = 25$
5. all the rows are covered, therefore $\Delta_A^+(x) = \emptyset$ and the algorithm terminates

Now $f_A = \sum_{j \in R} \theta_j = 50$ and the approximation holds: $f_A \leq (\ln|R| + 1)\, f^* \approx 2.79\, f^*$