Heuristic Algorithms for Combinatorial Optimisation problems

Ph.D. course in Computer Science

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Recombination heuristics

Constructive and exchange heuristics manage one solution at a time
(except for the *Ant System*)

Recombination heuristics manage several solutions in parallel

- start from a set (population) of solutions (individuals) obtained somehow
- recombine the individuals generating a new population

Their original aspect is the use of operations working on several solutions,
but they often include features of other approaches (sometimes renamed)

Some are nearly or fully deterministic

- Scatter Search
- Path Relinking

others are strongly randomised (often based on biological metaphors)

- genetic algorithms
- memetic algorithms
- evolution strategies

Of course the effectiveness of a method does not depend on the metaphor

General scheme

The basic idea is that

- good solutions share components with the global optimum
- different solutions can share different components
- combining different solutions, it is possible to merge optimal components more easily than building them step by step

The typical scheme of recombination heuristics is

- build a starting population of solutions
- as long as a suitable termination condition does not hold
- at each iteration (generation) update the population
  - extract single individuals and apply exchange operations to them
  - extract subsets of individuals (usually, pairs) and apply recombination operations to them
  - collect the individuals thus generated and choose whether to accept or not each of them and how many copies into the new population

Scatter Search

Scatter Search (*SS*), proposed by Glover (1977), proceeds as follows

1. generate a starting population of solutions
2. improve all of them with an exchange procedure
3. build a *reference set* $R = B \cup D$ where
   - subset B includes the best known solutions
   - subset D includes the "farthest" solutions (from B and each other)
     (this requires a distance definition, e.g. the Hamming distance)
4. for each pair of solutions $(x, y) \in B \times (B \cup D)$
   - "recombine" x and y, generating z
   - improve z obtaining z' with an exchange procedure
   - if $z' \notin B$ and B contains a worse solution, replace it with z'
     (we want no duplicates in the reference set)
   - if $z' \notin D$ and D includes a closer solution, replace it with z'
     (we want no duplicates in the reference set)
5. terminate when R is unchanged

The rationale is that

- the recombinations in $B \times B$ intensify the search
- the recombinations in $B \times D$ diversify the search

General scheme of the *Scatter Search* approach

Algorithm ScatterSearch(I, P, n_B, n_D)
$B := \emptyset$; $D := \emptyset$;
Repeat
  Stop $=$ *true*;
  For each $x \in P$ *do*
    $z :=$ SteepestDescent(I, x); *If* $f(z) < f(x^*)$ *then* $x^* := z$;
    $y_B := \arg\max_{y \in B} f(y)$; $y_D := \arg\min_{y \in D} d(y, B \cup D \setminus \{y\})$;
    If $z \notin B$ *and* $(|B| < n_B$ *or* $f(z) < f(y_B))$ *then*
      { B keeps the n_B best unique solutions }
      $B := B \cup \{z\}$; Stop := *false*; *If* $|B| > n_B$ *then* $B := B \setminus \{y_B\}$;
    ElseIf $z \notin D$ *and* $(|D| < n_D$ *or* $d(z, B \cup D \setminus \{y_D\}) > d(y_D, B \cup D \setminus \{y_D\}))$ *then*
      { D keeps the n_D most diverse unique solutions }
      $D := D \cup \{z\}$; Stop := *false*; *If* $|D| > n_D$ *then* $D := D \setminus \{y_D\}$;
    EndIf
  EndFor
  $P := \emptyset$;
  For each $(x, y) \in B \times (B \cup D)$ *do* { Recombine to build the new population }
    $P := P \cup$ Recombine(x, y, I);
  EndFor
until Stop $=$ *true*;
Return $(x^*, f(x^*))$;

Recombination procedure

The recombination procedure depends on the problem

Usually, solutions x and y are manipulated as subsets

1. include in z all the elements shared by x and y:

$$z := x \cap y$$

(*both solutions concur in suggesting those elements*)

2. augment solution z adding elements from $x \setminus z$ or $y \setminus z$
   - chosen at random or with a greedy selection criterium
   - alternatively from each source or freely from the two sources

   (*this is similar to a restricted constructive heuristic*)
3. if necessary, add external elements from $B \setminus (x \cup y)$
4. if subset z is unfeasible, apply an auxiliary exchange heuristic to make it feasible (repair procedure)

Examples

MDP

- start with $z := x \cap y$
- augment z with $k - |z|$ random or greedy points from $x \setminus z$ or $y \setminus z$
- no repair procedure is required

Max-SAT

- start with $z := x \cap y$
- augment z with $n - |z|$ random or greedy truth assignments from $x \setminus z$ or $y \setminus z$
- no repair procedure is required

Examples

KP

- start with $z := x \cap y$
- augment z with random or greedy elements from $x \setminus z$ or $y \setminus z$ respecting the capacity
- no repair procedure is required
- the solution could be augmented with elements from $B \setminus (x \cup y)$

SCP

- start with $z := x \cap y$
- augment z with random or greedy columns from $x \setminus z$ or $y \setminus z$ (avoiding the redundant ones)
- remove the redundant columns with a destructive phase

Path Relinking

Path Relinking (*PR*), proposed by Glover (1989), is generally used as a final intensification procedure more than as a stand-alone method

Given a neighbourhood N and an exchange heuristic based on it

- collect in a reference set R the best solutions generated by the auxiliary heuristic (elite solutions)
- for each pair of solutions x and y in R
  - build a path γ_{xy} from x to y in the search space of neighbourhood N applying to $z^{(0)} = x$ the auxiliary exchange heuristic, but choosing at each step the solution closest to the destination y

$$z^{(k+1)} := \arg \min_{z \in N(z^{(k)})} d(z, y)$$

    where d is a suitable metric function on the solutions
    In case of equal distance, optimise the objective function f
  - find the best solution z^*_{xy} along the path (and improve it)

$$z^*_{xy} := \arg \min_{k \in \{1,\ldots,|\gamma_{xy}|-1\}} f(z^{(k)})$$

  - if $z^*_{xy} \notin R$ and is better than the worst in R, add it to R

General scheme of the *Path Relinking* approach

Algorithm PathRelinking(I, P, n_R)

Repeat

$R := \emptyset$;

For each $x \in P$ *do*

$z :=$ SteepestDescent(I, x); *If* $f(z) < f(x^*)$ *then* $x^* := z$;

$y_R := \arg\max_{y \in R} f(y)$;

If $z \notin R$ *and* ($|R| < n_R$ *or* $f(z) < f(y_R)$) *then*

{ R keeps the n_R best unique solutions }

$R := R \cup \{z\}$; Stop := *false*; *If* $|R| > n_R$ *then* $R := R \setminus \{y_R\}$;

EndIf

EndFor

$P := \emptyset$;

For each $x \in R$ *and* $y \in R \setminus \{x\}$ *do* { Recombine to build the new population }

$z := x$; $z^* := x$;

While $z \neq y$ *do* { Build a path from x to y }

$Z := \arg\min_{z' \in N(z)} d(z', y)$; $z := \arg\min_{z' \in Z} f(z')$;

If $f(z) < f(z^*)$ *then* $z^* := z$

EndWhile;

If $z^* \notin P$ *then* $P := P \cup \{z^*\}$;

EndFor

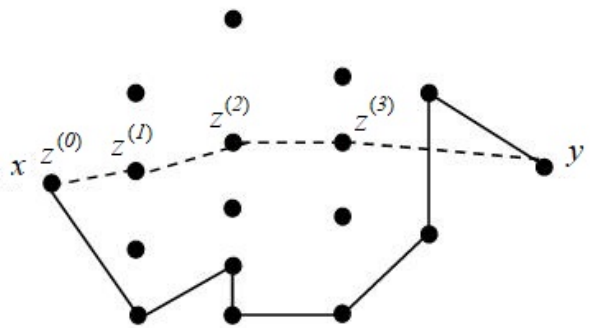
until Stop = *true*;

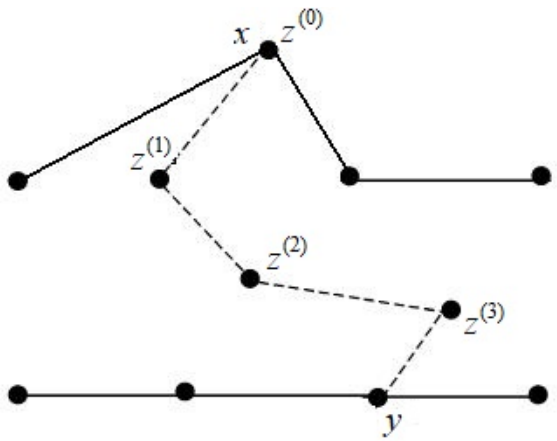
Return $(x^*, f(x^*))$;

Relinking paths

The paths explored in this way

- intensify the search, because they connect good solutions
- diversify the search, because they follow different paths with respect to the exchange heuristic (especially if the extremes are far away)





- since the distance of $z^{(k)}$ from y is decreasing, one can explore
  - worsening solutions without the risk of cyclic behaviours
  - unfeasible subsets without the risk of not getting back to feasibility

(*they do not improve directly, but open the way to improvements*)

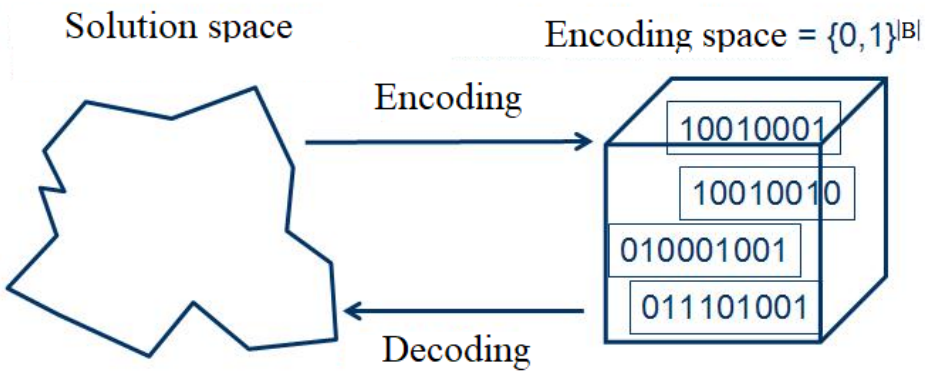
Variants

Given two solutions x and y, Path Relinking has several variants:

- *forward path relinking*: build a path from the worse to the better one
- *backward path relinking*: build a path from the better to the worse one
- *back-and-forward path relinking*: build both paths
- *mixed path relinking*: build a path with alternative steps from each extreme (updating the destination)
- *truncated path relinking*: build only the first steps of the path (if the good solutions are experimentally close to each other)
- *external path relinking*: build a path from one moving away from the other (if the good solutions are experimentally far from each other)

Encoding-based algorithms

Many recombination heuristics define and manipulate encodings of the solutions (i.e., compact representations), rather than the solutions



The aims of this approach are

- abstraction: conceptually distinguishing the method from the problem to which it is applied
- generality: build operators effective on every problem represented with a given family of encodings

In a strict sense, every representation of a solution in memory is an encoding: the term "encoding" tends to be used for the more involved and compact ones

The difference is blurred

Genetic algorithm

The genetic algorithm, proposed by Holland (1975), is the most famous

It builds and encodes a population $X^{(0)}$, and repeatedly applies:

1. selection: generate a new population starting from the current one
2. crossover: recombine subsets of two or more individuals
3. mutation: modify the individuals

Algorithm GeneticAlgorithm$\left(I, X^{(0)}\right)$

$\Xi := \mathit{Encode}\left(X^{(0)}\right)$; $x^* := \arg \min_{x \in X^{(0)}} f(x)$; { Best solution found so far }

For $g = 1$ *to* n_g *do*

$\Xi := \mathit{Selection}(\Xi)$;

$\Xi := \mathit{Crossover}(\Xi)$;

$x_c := \arg \min_{\xi \in \Xi} f(x(\xi))$;

If $f(x_c) < f(x^*)$ *then* $x^* := x_c$;

$\Xi := \mathit{Mutation}(\Xi)$;

$x_m := \arg \min_{\xi \in \Xi} f(x(\xi))$;

If $f(x_m) < f(x^*)$ *then* $x^* := x_m$;

EndFor;

Return $(x^*, f(x^*))$;

Features of a good encoding

The performance of a genetic algorithm depends on the encoding

The following properties should be satisfied (with decreasing importance)

1. each solution should have an encoding, except for dominated ones;
   otherwise, there would be unreachable solutions
2. different solutions should have different encodings
   (or the best solution with a given encoding should be easy to find);
   otherwise, there would be unreachable solutions
3. each encoding should correspond to a feasible solution;
   otherwise, the population would include useless individuals
4. each solution should correspond to the same number of encodings;
   otherwise, some solutions would be unduly favoured
5. the encoding and decoding operations should be efficient,
   otherwise, the algorithm would be inefficient
6. locality: small changes to the encoding should induce small changes to the solution, otherwise intensification and diversification would be impossible

These conditions depend very much on the constraints of the problem

(*so much for abstraction...*)

Feasible and unfeasible encodings

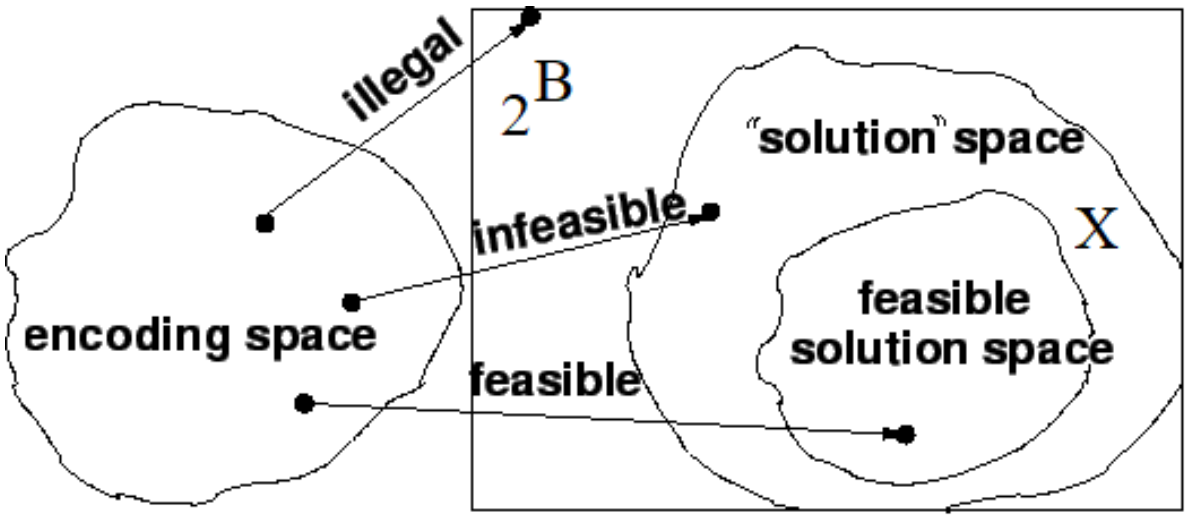
Mutation and crossover operators easily produce unfeasible subsets
if property 3 is not satisfied; this may imply the violation of

1. quantitative constraints (e.g., a capacity is exceeded)
2. structural constraints (e.g., the solution is not made of circuits)

The distinction is conventional

The second kind of infeasibility is harder to repair, because
it concerns constraints that interact more strongly with each other

Some encodings guarantee structural (though not quantitative) feasibility



Encodings: the incidence vector

The most direct encoding for Combinatorial Optimisation problems is the binary incidence vector $\xi \in \mathbb{B}^{|B|}$

$$\begin{cases} \xi_i = 1 \text{ indicates that } i \in x \\ \xi_i = 0 \text{ indicates that } i \notin x \end{cases}$$

A generic binary vector corresponds

- in the *KP* to a set of objects: its weight could be excessive
- in the *SCP* to a set of columns: it could leave uncovered rows
- in the *PMSP* and in the *BPP* to a set of assignments of tasks (objects) to machines (containers): it could make zero or more assignments for an element; in the *BPP*, it could violate the capacity of some container;
- in the *TSP* to a set of arcs: it could not form a Hamiltonian circuit
- in the *CMSTP* (*VRP*) to a set of edges (arcs): it could not form a tree (set of cycles), or exceed the capacity of the subtrees (circuits)

Encodings: symbolic strings

If the ground set is partitioned into components

(*objects, tasks, Boolean variables, vertices, nodes...*)

$$B = \bigcup_{c \in C} B_c \quad \text{with } B_c \cap B_{c'} = \emptyset \text{ for each } c \neq c'$$

and the feasible solutions contain one element of each component

$$|x \cap B_c| = 1 \text{ for each } c$$

one can

- define for each $c \in C$ an alphabet of symbols describing component B_c
- encode the solution into a string of symbols $\xi \in B_1 \times \ldots B_{|C|}$

$$\xi_c = \alpha \text{ indicates that } x \cap B_c = \{(c, \alpha)\}$$

Examples of encodings:

- *Max-SAT*: a string of n Boolean values, one for each logical variable
- *PMSP*: a string of machine labels, one for each task
- *BPP*/*CMSTP*: a string of container/subtree labels, one for each object/vertex:
  - the structural constraint on object assignment is enforced
  - the quantitative constraint on capacity is neglected
- for the *VRP*, a string of vehicle labels, one for each node (but capacity is neglected and decoding the circuit for each vehicle is an $\mathcal{NP}$-complete problem)
- the solutions of the *TSP*, the *KP*, the *SCP* are not partitions

Encodings: permutations of a set

A common encoding is given by the permutations of a set

- if the solutions are permutations, this is the natural encoding
  (*TSP* solutions are subsets of arcs, but also permutations of nodes)
- if the solutions are partitions and the objective is additive on the subsets, the *order-first split-second* method transforms permutations into partitions
  (*but solutions and encodings do not correspond one-to-one*!)
- if the problem admits a constructive algorithm that at each step
  1. chooses an element
  2. chooses how to add it to the solution (if many ways exist)

  we can feed elements to the algorithm following the permutation
  (*depending on step* 2, *some solutions could have no encoding*)

Selection

At each generation g a new population $\Xi^{(g)}$ is built extracting $n_p = |\Xi^{(g)}|$ individuals from the current population $\Xi^{(g-1)}$

$$\Xi^{(g)} := \text{Selection}\left(\Xi^{(g-1)}\right);$$

The extraction follows two fundamental criteria

1. an individual can be extracted more than once
2. better individuals are extracted with higher probability

$$\varphi(\xi) > \varphi(\xi') \Rightarrow \pi_\xi \geq \pi_{\xi'}$$

where the fitness $\varphi(\xi)$ is a measure of the quality of individual ξ

- for a maximisation problem, commonly

$$\varphi(\xi) = f(x(\xi))$$

- for a minimisation problem, commonly

$$\varphi(\xi) = UB - f(x(\xi))$$

where $UB \geq f^*$ is a suitable upper bound on the optimum

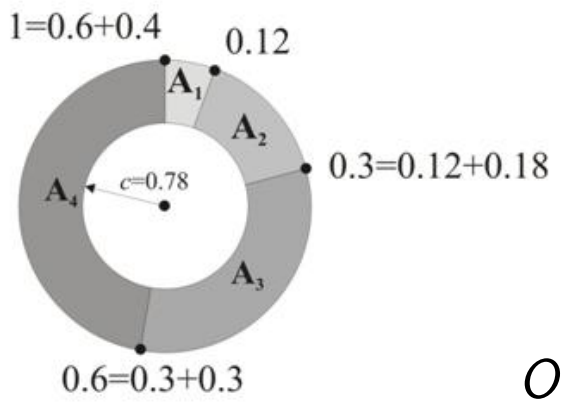
Proportional selection

The original scheme proposed by Holland (1975) assumed a probability proportional to fitness

$$\pi_\xi = \frac{\varphi(\xi)}{\sum_{\xi \in \Xi} \varphi(\xi)}$$

This is named roulette-wheel selection or spinning wheel selection:

- given the fitness for $\xi_i \in \Xi$ $(i = 1, \ldots, n_p)$
- build the intervals $\Gamma_i = \left(\sum_{k=1}^{i-1} \pi_{\xi_i}; \sum_{k=1}^{i} \pi_{\xi_i} \right]$ in $O(n_p)$ time
- extract a random number $r \in U(0; 1]$
- choose individual i^* such that $r \in \Gamma_{i^*}$ in $O(\log n_p)$ time each



Overall $O(n_p \log n_p)$ *time*

Rank selection

The proportional selection suffers from

- stagnation: in the long term, all individuals tend to have a good fitness, and therefore similar selection probabilities
- premature convergence: if the best individuals are bad and the other ones very bad, the selection quickly generates a bad population

To overcome these limitations, one should at the same time

- assign different probabilities to the individuals
- limit the difference of probability among the individuals

The rank selection method

- sorts the individuals by nondecreasing fitness

$$\Xi^{(g)} = \left\{\xi_1, \ldots, \xi_{n_p}\right\} \text{ with } \varphi_{\xi_1} \leq \ldots \leq \varphi_{\xi_{n_p}}$$

- assigns to the k-th individual a probability equal to

$$\pi_{\xi_k} = \frac{k}{\sum_{k=1}^{n_p} k} = \frac{2k}{n_p\left(n_p - 1\right)}$$

It can be done in $O\left(n_p \log n_p\right)$ time as in the previous case

Tournament selection

An efficient compromise consists in

- extracting n_p random subsets $\Xi_1, \ldots, \Xi_{n_p}$ of size α
- selecting the best individual from each subset

$$\xi_k := \arg\max_{\xi \in \Xi_k} \varphi(\xi) \qquad k = 1, \ldots, n_p$$

in time $O(n_p \alpha)$

Parameter α tunes the strength of the selection:

- $\alpha \approx n_p$ favours the best individuals
- $\alpha \approx 2$ leaves chances to the bad individuals

All selection procedures admit an elitist variant, which
includes in the new population the best individual of the current one

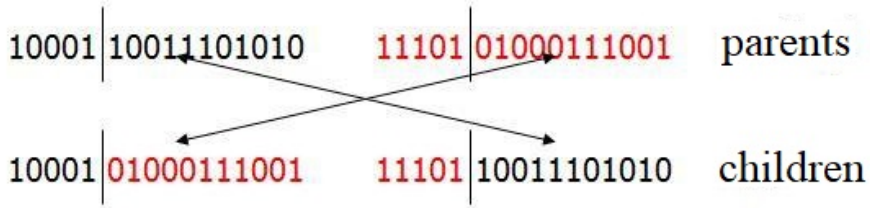
(*always keep the best individual found so far*)

Crossover

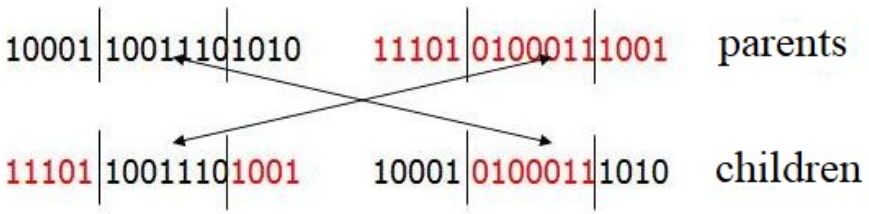
The crossover operator combines $k \geq 2$ individuals to generate other k

The most common ones set $k = 2$ and are

- simple crossover:
  - extract a random position with uniform probability
  - split the encoding in two parts at the extracted position
  - exchange the final parts of the encodings of the two individuals



- double crossover:
  - extract two positions at random with uniform probability
  - split the encoding in three parts at the extracted positions
  - exchange the extreme parts of the encodings of the two individuals



Crossover

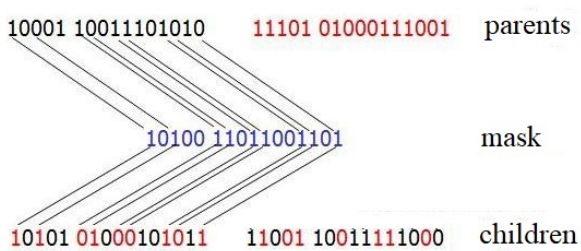
Generalizing, one obtains the

- α points crossover:
  - extract α positions at random with uniform probability
  - split the encoding in $\alpha + 1$ parts at the extracted positions
  - exchange the odd parts of the encodings of the two individuals (first, third, etc. . .)

For small values of α, this implies a positional bias:
symbols close in the encoding tend to remain close

To cancel this bias, one can adopt the

- uniform crossover:
  - build a random binary vector $m \in U(\mathbb{B}^n)$ ("mask")
  - if $m_i = 1$ exchange the symbols in position i of the two individuals, if $m_i = 0$ keep them unmodified



Crossover versus Scatter Search and Path Relinking

The crossover operator resembles the recombination phase of *SS* and *PR*

The main differences are that

1. it recombines the symbols of the encodings, instead of
   - recombining the solutions (*SS*)
   - performing a chain of exchanges on the solutions (*PR*)
2. it operates on the whole population, instead of only a reference set R
3. it operates on random pairs of individuals, instead of methodically scanning all pairs of solutions of R
4. it generates a pair of new individuals, instead of
   - generating a single intermediate solution (*SS*)
   - visiting the intermediate solutions and choosing the best one (*PR*)
5. the new individuals enter the population directly, instead of becoming candidates for the reference set

However, classifying an operator can be a matter of taste

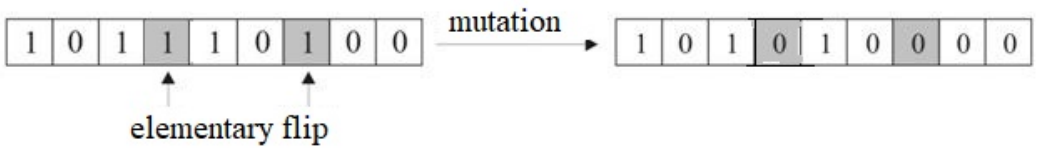
Mutation

The mutation operator modifies an individual to generate a similar one

- scan encoding ξ one symbol at a time
- decide with probability π_m to modify the current symbol

The kind of modification usually depends on the encoding

- binary encodings: flip ξ_i into $\xi_i' := 1 - \xi_1$



- symbol strings: replace ξ_c with a random symbol $\xi_c' \in B_c \setminus \{\xi_c\}$ selected with a uniform probability
- permutations: there are many proposals
  - exchange two random elements in the permutation (*swap*)
  - reverse the stretch between two random positions of the permutation
  - ...

Mutation versus exchange heuristics

The mutation operator has strong relations with exchange operations

The main differences are that

1. it modifies the symbols of an encoding,
   instead of exchanging elements of a solution
2. it operates on random symbols,
   instead of exploring a neighbourhood systematically
3. it operates on a random subset of symbols of size unknown *a priori*,
   somewhat like sampling a very large scale neighbourhood,
   instead of exchanging a fixed number of elements
4. it operates on random individuals, instead of all solutions
5. the new individuals enter the population directly,
   instead of becoming candidates for the reference set

However, classifying an operator can be a matter of taste

The feasibility problem

If the encoding is not fully invertible, crossover and mutations sometimes generate encodings that do not correspond to feasible solutions

We distinguish between

- feasible encodings that correspond to feasible solutions
- unfeasible encodings that correspond to legal, but unfeasible subsets

The existence of unfeasible encodings implies several disadvantages:

- inefficiency: computational time is lost handling meaningless objects
- ineffectiveness: the heuristic explores less solutions (possibly, none)
- design problems: fitness must be defined also on unfeasible subsets

There are three main approaches to face this problem

1. special encodings and operators (*avoid or limit infeasibility*)
2. repair procedures (*turn infeasibility into feasibility*)
3. penalty functions (*accept infeasibility, but discourage it*)

Special encodings and operators

The idea is to investigate

- encodings that (nearly) always yield feasible solutions, such as
  - permutation encodings and order-first split-second decodings for partition problems (*CMSTP*, *VRP*, etc. . .)
  - permutation encodings and constructive heuristic decodings for scheduling problems (*PMSP*,. . .)
- crossover and mutation operators that maintain feasibility, such as
  - specialised operators (*Order* or *PMX* crossover for the *TSP*)
  - operators that simulate moves on solutions (k-exchanges)

These methods

- tend to closely approximate exchange and recombination heuristics based on the concept of neighbourhood
- give up the idea of abstraction and focus on the specific problem, contrary to classical genetic algorithms

Repair procedures

A repair procedure is a refined decoder function $x_R : \Xi \to X$ that

- decodes any encoding ξ into a possibly unfeasible solution $x(\xi) \notin X$
- transforms subset $x(\xi)$ into a feasible solution $x_R(\xi) \in X$
- returns $x_R(\xi)$

The procedure is applied to each unfeasible encoding $\xi \in \Xi^{(g)}$

- in some methods, the encoding $\xi(x_R(\xi))$ replaces ξ in $X^{(g)}$
- in other ones, ξ remains in $\Xi^{(g)}$ and $x_R(\xi)$ is used only to update x^*

The methods of the first family

- maintain a population of feasible solutions

but they introduce

- a strong bias in favour of feasible encodings
- a bias in favour of the feasible solutions most easily obtained with the repair procedure

Penalty functions: measuring the infeasibility

If the objective function is extended to unfeasible subsets $x \in 2^B \setminus X$, the fitness function $\phi(\xi)$ can be extended to any encoding, but many unfeasible subsets have a fitness larger than the optimal solution

The selection operator tends to favour such unfeasible subsets

To avoid that, the fitness function must combine

- the objective value $f(x(\xi))$
- a measure of infeasibility $\psi(x(\xi))$

$$\begin{cases} \psi(x(\xi)) = 0 \text{ if } x(\xi) \in X \\ \psi(x(\xi)) > 0 \text{ if } x(\xi) \notin X \end{cases}$$

If the constraints of the problem are espressed by equalities or inequalities, $\psi(x)$ can be defined as a weighted sum of their violations

How to define the weights?
Are they fixed, variable or adaptive?

Penalty functions: definition of the fitness

The most typical combinations are

- absolute penalty: minimise ψ and f lexicographically;
  given two encodings ξ and ξ' in a rank or tournament selection
  - choose the less unfeasible one
  - if they are equally unfeasible (e. g., both feasible), choose the better
- proportional penalty: use a linear combination of f and ψ

$$\varphi(\xi) = f(x(\xi)) - \alpha\psi(x(\xi)) + M \quad \text{for maximisation problems}$$

$$\varphi(\xi) = -f(x(\xi)) - \alpha\psi(x(\xi)) + M \quad \text{for minimisation problems}$$

  where $\alpha > 0$ and offset M guarantees $\varphi(\xi) \geq 0$ for all encodings
- penalty obtained by repair, that is keep the unfeasible encoding,
  but derive its fitness from the objective value of the repaired solution

$$\varphi(\xi) = f(x_R(\xi)) \text{ or } \varphi(\xi) = UB - f(x_R(\xi))$$

  since usually $f(x_R(\xi))$ is worse than $f(x(\xi))$

Proportional penalty functions: weight tuning

Experimentally, it is better to use the smallest effective penalty

- if the penalty is too small, too few feasible solutions are found
- if the penalty is too large, the search is confined within a part of the feasible region (*"hidden" feasible solutions are hard to find*)

A good value of the parameter α tuning the penalty can be found with

- dynamic methods: increase α over time according to a fixed scheme (*first reach good subsets, then enforce feasibility*)
- adaptive methods: update α depending on the situation
  - increase α when unfeasible encodings dominate the population
  - decrease α when feasible encodings dominate
- evolutionary methods: encode α in each individual, in order to select and refine both the solution and the algorithm parameter

Memetic algorithms

Memetic algorithms (Moscato, 1989) are inspired by the concept of meme (Dawkins, 1976) that is a basic unit of reproducible cultural information

- genes are selected only at the phenotypic expression level
- memes also adapt directly, as in Lamarckian evolution

Out of the metaphor, memetic algorithms combine

- "genotypic" operators that manipulate the encodings (crossover and mutation)
- "phenotypic" operators that manipulate the solutions (local search)

In short, the solutions are improved with exchanges before reencoding

Several parameters determine how to apply local search

- how often (at every generation, or after a sufficient diversification)
- to which individuals (all, the best ones, the most diversified ones)
- for how long (until a local optimum, beyond, or stopping before)
- with what method (steepest descent, VNS, ILS, etc...)

Evolution strategies

They have been proposed by Rechenberg and Schwefel (1971)

The main differences with respect to classical genetic algorithms are:

- the solutions are encoded into real vectors
- a small population of μ individuals generate λ candidate descendants (*originally*, $\mu = 1$)
- the new individuals compete to build the new population
  - in the (μ, λ) strategy the best μ descendants replace the original population, even if some are dominated
  - in the $(\mu + \lambda)$ strategy the best μ individuals overall (predecessors or descendants) survive in the new population
- the mutation operator sums to the encoding a random noise with a normal distribution of zero average

$$\xi' := \xi + \delta \text{ with } \delta \in N(0, \sigma)$$

- originally, the crossover operator was not used (*now it is*)