Decision Methods and Models

Master's Degree in Computer Science

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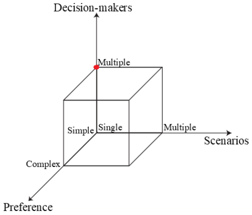
Group decisions

We assume

- a certain environment: $|\Omega| = 1 \Rightarrow f(x, \bar{\omega})$ reduces to $f(x)$
- multiple decision-makers: $|D| > 1$
- preference relations Π_d that are weak orders



We consider the case in which

- the final choice must be taken by the whole set D
- the decision-makers must agree before taking it

The axiomatic approach

All historical attempts to build a social welfare function have failed to guarantee all desirable properties

The axiomatic approach

- lists the desired properties
- tries to design a function that satisfies them by construction

In 1950 Kenneth Arrow proved this to be impossible

The sketch of the proof is interesting to understand the basic problems

Of course, it will require some terminology and notation

Preference estrictions and profiles

Given a preference relation $\Pi_d(X) \subseteq X \times X$ and a subset $X' \subseteq X$, the restriction $\Pi_d(X')$ of $\Pi_d(X)$ to X' is the preference relation

$$\Pi_d(X') = \Pi_d(X) \cap (X' \times X')$$

Just remove from $\Pi_d(X)$ all pairs (x, y) with $x \notin X'$ or $y \notin X'$

Example

Let $X = \{a, b, c\}$ and $\Pi_d(X) = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$

If $X' = \{a, b, c\}$, the restriction simply becomes

$$\Pi_d(X') = \{(a, a), (a, c), (c, c)\}$$

A preference profile $\Pi(X) \in \mathcal{D}(X)^{|D|}$ is a vector of $|D|$ weak orders on X (one for each individual)

Its restriction $\Pi(X')$ is the vector composed by the restrictions $\Pi_d(X')$

Comparing preference profiles

Two profiles $\Pi(X)$ and $\Pi'(X)$ can be compared on a solution pair (x, y)

Π' promotes x over y more than Π when

1. every individual d who strictly prefers x in Π also prefers x in Π'

$$x \prec_{\Pi,d} y \Rightarrow x \prec_{\Pi',d} y$$

2. every individual d who is indifferent between x and y in Π weakly prefers x in Π'

$$x \sim_{\Pi,d} y \Rightarrow x \preceq_{\Pi',d} y$$

So, the preference for x increases monotonically if Π is replaced by Π'

Example

Consider $D = \{d_1, d_2, d_3, d_4\}$ and $X = \{x, y, z\}$

	Π	Π'
d_1	$z \prec x \prec y$	$z \sim x \prec y$
d_2	$x \sim y \prec z$	$x \sim y \prec z$
d_3	$y \prec x \prec z$	$y \prec z \prec x$
d_4	$z \prec x \sim y$	$z \prec x \prec y$

In this situation, Π' promotes more than Π

- x over y, but not y over x because
  - d_1 keeps a strict preference
  - d_2 keeps an indifference
  - d_4 changes from indifference to strict preference
- both y over z and z over y because
  - the preferences are unchanged
- neither x over z nor z over x because
  - d_1 changes from strict preference to indifference
  - d_3 reverses a strict preference

Dictator

A dictator is an individual $s \in D$ such that

$$x \prec_{\Pi,s} y \Rightarrow x \prec_{\Pi,D} y$$

for every solution pair $x, y \in X$ and for every preference profile $\Pi \in \mathcal{D}^{|D|}$

In words, every weak preference of the dictator translates
into a weak preference of the group through the social welfare function

The existence of a dictator is a property of the social welfare function

This concept can be relaxed and generalised in two ways

1. replacing the dictator $s \in D$ with a subset of individuals
2. restricting the absolute power to a single pair of solutions (x, y)

A decisive set for solution pair (x, y) is a subset of individuals $S \subseteq D$ such that

$$x \prec_{\Pi,d} y \text{ for all } d \in S \Rightarrow x \prec_{\Pi,D} y$$

A specific weak unanimous preference of the decisive set translates
into a weak preference of the group through the social welfare function

We could roughly name it an oligarchy

Arrow's axioms

In 1950 Kenneth Arrow summarised the desirable properties of a social welfare function into the following set of axioms

1. nontriviality: there are at least three alternatives and two individuals

   $$|X| \geq 3 \qquad |D| \geq 2$$

   Condorcet method solves satisfactorily the case with two alternatives
2. universality: $g(\Pi)$ is defined for all $\Pi \in \mathcal{D}^{|D|}$
   The social welfare function must solve the problem for all profiles
3. weak order: $g(\Pi) \in \mathcal{D}$ for all $\Pi \in \mathcal{D}^{|D|}$
   The social welfare function must return a weak order for all profiles

   The Condorcet method violates this axiom
4. independence from irrelevant alternatives:

   $$g(\Pi(X)) = \Pi_D(X) \Rightarrow g(\Pi(X')) = \Pi_D(X') \text{ for all } X' \subseteq X$$

   The social welfare function on a restricted alternative set
   is the restriction of the original group preference

   The Borda method and the plurality system violate this axiom

Arrow's axioms

In 1950 Kenneth Arrow summarised the desirable properties of a social welfare function into the following set of axioms

5. monotony: given two alternatives $x, y \in X$
   and two preference profiles $\Pi(X)$ and $\Pi'(X)$ such that
   - $\Pi'(X)$ promotes x over y more than $\Pi(X)$
   - $\Pi(X)$ have the same preferences on all pairs not including x
   - $x \prec_{\Pi,D} y$

   Then, $x \prec_{\Pi',D} y$

   The social welfare functions maintains a preference for x over y if x is further promoted
6. popular sovereignty: the codomain of $g(\cdot)$ is $\mathcal{D}(X)$

$$\exists \Pi(X) \in \mathcal{D}(X)^{|D|} : g(\Pi(X)) = \bar{\Pi} \text{ for all } \bar{\Pi} \in \mathcal{D}(X)$$

   Every weak order can be obtained choosing $|D|$ suitable preferences

   Function $g(\cdot)$ is surjective
7. nondictatorship: no individual is a dictator

   The lexicographic method violates this axiom

Arrow's axioms (modern version)

The modern version of Arrow's theorem adopts a weaker axiom

$5 - 6'$) unanimity (or Pareto efficiency):

if all individuals agree on a preference, the group also agrees

$$x \prec_{\Pi,d} y \text{ for all } d \in D \Rightarrow x \prec_D y$$

to replace monotony (5) and popular sovereignty (6)

It is weaker because

- popular sovereignty guarantees the existence of a profile Π' such that $x \prec_{\Pi',D} y$
- the unanimous profile Π for which $x \prec_{\Pi,d} y$ for all $d \in D$ promotes x over y more than Π'
- monotony guarantees that $x \prec_{\Pi',D} y \Rightarrow x \prec_{\Pi,D} y$

Theorem

Any social welfare function satisfying axioms 1 to 6 implies a dictator

The proof goes through the following stages

1. there is a decisive set for a specific pair of solutions
2. this decisive set can be reduced to a single individual
3. the individual is decisive for every pair of solutions

The steps are based on applying the axioms to special cases
and drawing general conclusions

While correct, they look rather counterintuitive

The discussion focuses on how to interpret the model

Sketch of the proof (1)

Consider any social welfare function $g(\cdot)$

By unanimity, set D is decisive for ever pair of alternatives

Given a specific pair (w, z), is there a smaller decisive set $V = D \setminus \{d\}$?

- by progressively removing from V individuals that do not affect this, generate a minimal decisive set for (w, z)

Is there a smaller set $V \setminus \{d\}$ that is decisive for another pair (x, y)?

- remove from V further individuals, possibly changing the pair, until
  - set V is decisive for (x, y)
  - no proper subset of V is decisive for any pair of alternatives

By universality, function $g(\cdot)$ applies to any preference profile

So, we can challenge it to solve any special case

Sketch of the proof (2)

By nontriviality, consider a problem with $|X| \geq 3$ and $|D| \geq 2$ and split D into

- an arbitrary individual d from the decisive set V
- the rest of the decisive set, $V \setminus \{d\}$
- the rest of the decision-maker set, $D \setminus V$

It is possible that V and D coincide and $D \setminus V$ is empty

What about $V \setminus \{d\}$?

If $V \setminus \{d\}$ is empty, $V = \{d\}$ and individual d is decisive for pair (x, y)

By contradiction, assume $V \setminus \{d\} \neq \emptyset$ and the following preference profile

$\{d\}$	$V \setminus \{d\}$	$D \setminus V$
x	z	y
y	x	z
z	y	x
$\ldots$	$\ldots$	$\ldots$

where the preferences of all individuals on the other alternatives can be ignored by independence from irrelevant alternatives

Sketch of the proof (2)

$\{d\}$	$V \setminus \{d\}$	$D \setminus V$
x	z	y
y	x	z
z	y	x
...	...	...

Notice that

- V is decisive for (x, y) and $x \prec_{d'} y$ for all $d' \in D \;\Rightarrow\; x \prec_D y$
- all members of $V \setminus \{d\}$ think that $z \prec y$,
  while and all other individuals think the opposite
  What is the group preference on (y, z)?
- if we assume that $z \prec_D y$, then $V \setminus \{d\}$ is decisive for (y, z)
- since V is a minimal decisive set, this is not possible $\Rightarrow y \preceq_D z$
- by weak order, $x \prec_D y$ and $y \preceq_D z \Rightarrow x \prec_D z$
- this is also impossible, because d would be decisive for (x, z),
  as the only individual preferring x to z

So, we go back and conclude that $V = \{d\}$ and d is decisive for (x, y)

Sketch of the proof (3)

Now apply function $g(\cdot)$ to another preference profile

$\{d\}$	$D \setminus \{d\}$
x	y
y	w
w	x
$\dots$	$\dots$

Notice that

- by unanimity, $y \prec_{d'} w$ for all $d' \in D \Rightarrow y \prec_D w$
- since d is decisive for (x, y) and $x \prec_d y \Rightarrow x \prec_D y$
  even if all other individuals disagree
- by weak order, $x \prec_D y$ and $y \prec_D w \Rightarrow x \prec_D w$

But d is the only individual preferring x to w:
d is decisive for (x, w) for all $w \in X \setminus \{x\}$

Sketch of the proof (4)

Now apply function $g(\cdot)$ to another preference profile

$\{d\}$	$D \setminus \{d\}$
z	w
x	z
w	x
$\dots$	$\dots$

with w and z generic solutions different from x (but possibly equal to y)

Notice that

- by unanimity, $z \prec_{d'} x$ for all $d' \in D \Rightarrow z \prec_D x$
- since d is decisive for (x, w) and $x \prec_d w \Rightarrow x \prec_D w$
  even if all other individuals disagree
- by weak order, $z \prec_D x$ and $x \prec_D w \Rightarrow z \prec_D w$

But d is the only individual preferring z to w:
d is decisive for (z, w) for all $w, z \in X \setminus \{x\}$

Sketch of the proof (4)

Finally, apply function $g(\cdot)$ to a last preference profile

$\{d\}$	$D \setminus \{d\}$
z	w
w	x
x	z
$\dots$	$\dots$

with w and z generic solutions different from x (but possibly equal to y)

Notice that

- by unanimity, $w \prec_{d'} x$ for all $d' \in D \Rightarrow w \prec_D x$
- since d is decisive for (z, w) and $z \prec_d w \Rightarrow z \prec_D w$
  even if all other individuals disagree
- by weak order, $z \prec_D w$ and $w \prec_D x \Rightarrow z \prec_D x$

But d is the only individual preferring z to w:
d is decisive for (z, x) for all $z \in X \setminus \{x\}$

In summary, d is decisive for all pairs of alternatives: d is a dictator

Criticism to Arrow's axioms

Most of Arrow's axioms have been criticised in the following decades

1. nontriviality: of course, we need $|D| \geq 2$, but what if $|X| = 2$?

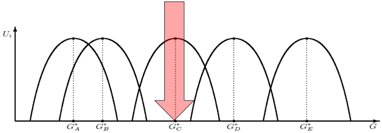
   Duncan Black showed that if there are only two alternatives, the Condorcet method satisfies all other axioms

   But how to reduce the alternatives to two? Bipartitism? Primaries?

   This just pushes the problem to the lower level

2. universality: do we really need $g(\cdot)$ to work in all cases?

   Black proved that the Condorcet method satisfies all axioms when X admits a linear order for which all preferences Π_d are "unimodal": following the order, the alternatives first progressively improve, then progressively worsen



   Moreover, sorting the individuals based on their best alternative, the best alternative of the median individual is the best of the group

Criticism to Arrow's axioms

Most of Arrow's axioms have been criticised in the following decades

3. weak order: can we accept incompleteness and/or intransitivity?
   As already observed, Condorcet circuits open the way to manipulations
4. independence from irrelevant alternatives:
   can we really ignore the other alternatives?

   It depends on the model: the position of two alternatives in the overall order might be related to their appeal to the individual

   Example

d_1	d_2
Coffee	Tea
Water	Coffee
Lemonade	Water
Cola	Lemonade
Tea	Cola

Coffee looks like a good alternative for both

Criticism to Arrow's axioms

d_1	d_2
Coffee	Tea
Water	Coffee
Lemonade	Water
Cola	Lemonade
Tea	Cola

While coffee looks like a good alternative for both individuals,
the restriction of the two preference relations to $X' = \{\text{Coffee}, \text{Tea}\}$ is

d_1	d_2
Coffee	Tea
Tea	Coffee

that looks quite different

A quantitative measure of utility could solve the problem, but would reintroduce the complexity of multiple-attribute utility theory

Criticism to Arrow's axioms

5. unanimity: should a unanimous preference always be applied?

   In the radical democracy modelled by Arrow, it should

   Many political ideologies exclude some decisions
   from the formal aggregation of individual preferences

   - "nonnegotiable values" for the Catholic church
   - human rights for the liberal thought
   - ...

   Of course, this opens the problem of how to define such decisions