Decision Methods and Models

Master's Degree in Computer Science

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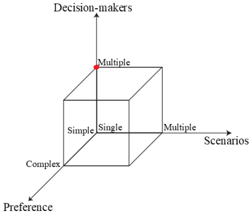
Group decisions

We assume

- a certain environment: $|\Omega| = 1 \Rightarrow f(x, \bar{\omega})$ reduces to $f(x)$
- multiple decision-makers: $|D| > 1$
- preference relations Π_d that are weak orders



We consider the case in which

- the final choice must be taken by the whole set D
- the decision-makers must agree before taking it

Basic assumptions

We assume

- a finite feasible region X
- a deterministic environment $(|\Omega| = 1)$
- an invertible impact function f,
  so that each preference relation Π_d can be directly imposed on X
- the decision-makers $d \in D$ are called individuals,
  or citizens, voters, agents, judges
- all preference relations Π_d are weak orders

These assumptions reduce the complexity to a central point

- how to aggregate the individual preferences Π_d
  into a group preference Π_D?

The weak order set

A preference relation on X is a subset of $X \times X$, i.e., an element of $2^{X \times X}$

Let $\mathcal{D}(X) \subset 2^{X \times X}$ be the set of all weak orders on X,
namely all preference relations that are reflexive, transitive and complete

For example, $\mathcal{D}(\{a, b, c\})$ includes 13 relations:

- 6 total orders corresponding to the permutations

$$\begin{array}{cc} a \prec b \prec c & a \prec c \prec b \\ b \prec a \prec c & b \prec c \prec a \\ c \prec a \prec b & c \prec b \prec a \end{array}$$

- 6 weak orders with ties

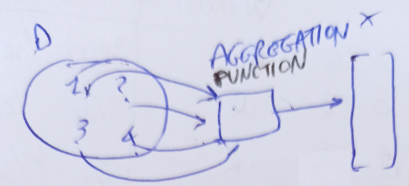
$$\begin{array}{ccc} a \prec b \sim c & b \prec a \sim c & c \prec a \sim b \\ a \sim b \prec c & a \sim c \prec b & b \sim c \prec a \end{array}$$

- 1 total indifference relation

$$a \sim b \sim c$$

Every individual $d \in D$ is associated with one of them

Social welfare function



A social welfare function $g : \mathcal{D}(X)^{|D|} \rightarrow 2^{X \times X}$ is a function that

- receives the preferences $\Pi_d \in \mathcal{D}(X)$ associated with the individuals (all weak orders by assumption)
- returns a preference relation $\Pi_D \in 2^{X \times X}$ for the whole group

First we discuss some historical proposals (*and their failures*)

Then we discuss an axiomatic approach (*and its failure*)

Aggregation problems

We have already discussed several aggregation problems

- several indicators into a single multi-attribute utility
- several scenarios into a single choice criterium or stochastic utility
- several individual preferences into a single group preference

They exhibit similar phenomena, but also specific differences; in fact

- aggregating indicators was very complex, but possible
- aggregating scenarios was impossible in conditions of ignorance
- aggregating scenarios was reasonably easy in conditions of risk

Aggregating individual preferences will be impossible

The Condorcet method

The Condorcet method (also known as simple majority method) performs a sort of election on each pair of alternatives

$$x \preceq_D x' \Leftrightarrow |\{d \in D : x \preceq_d x'\}| \geq |\{d \in D : x' \preceq_d x\}|$$

An alternative preferred by more individuals is preferred by the group

Indifferent individuals have no effect, as they are counted on both sides

The Condorcet method: limitation

Even if all individual preferences are weak orders, the Condorcet method does not guarantee that the group preference is a weak order

The problem concerns transitivity and is known as the Condorcet paradox

The classical example concerns three alternatives ($X = \{a, b, c\}$) with

- Π_1: $a \prec b \prec c$
- Π_2: $b \prec c \prec a$
- Π_3: $c \prec a \prec b$

The definition implies that

- $a \prec_D b$ because two individuals strictly prefer a over b
- $b \prec_D c$ because two individuals strictly prefer b over c
- $c \prec_D a$ because two individuals strictly prefer c over a

But then Π_D has a circuit of strict preferences

Historical examples in parliaments abound

(*they are solved fixing an agenda that forces an arbitrary choice*)

In general, all variants that solve the problem removing solutions fail

The Borda method

The Borda method builds a value function for each individual (Borda count)

$$B_d(x) = |\{x' \in X : x \preceq_d x'\}|$$

aggregates them with a simple suminto a group value function

$$B_D(x) = \sum_{d \in D} D_d(x)$$

and derives the group preference from the group value function

$$x \preceq_D x' \;\Leftrightarrow\; B_D(x) \geq B_D(x')$$

The group preference is a weak order by construction

The Borda method: example

Order	Individuals 1	2	3	4	5	6	7
1	a	b	c	a	b	c	a
2	b	c	d	b	c	d	b
3	c	d	a	c	d	a	c
4	d	a	b	d	a	b	d

This yields the following Borda count:

X	$B_D(x)$
a	$4+1+2+4+1+2+4=18$
b	$3+4+1+3+4+1+3=19$
c	$2+3+4+2+3+4+2=20$
d	$1+2+3+1+2+3+1=13$

and, consequently, the preference relation $c \prec b \prec a \prec d$

The Borda method: limitation

In the Condorcet method, a preference depends only on two alternatives

In the Borda method, the set of alternatives X appears in the definition: it affects the choice, and can be used to manipulate it

(X *is not always obvious a priori*)

This allows rank reversal, as in the *AHP*: the preference between two alternatives can depend on other irrelevant ones

Example

Let us remove d from the previous example

Order	Individuals 1	2	3	4	5	6	7
1	a	b	c	a	b	c	a
2	b	c	a	b	c	a	b
3	c	a	b	c	a	b	c

X	$B_D(x)$
a	$3+1+2+3+1+2+3=15$
b	$2+3+1+2+3+1+2=14$
c	$1+2+3+1+2+3+1=13$

This is because d never prevailed on c, but often on a

The plurality system

The plurality system builds a value function (as the Borda method): it counts the individuals that prefer each alternative to all other ones

$$V_D(x) = |\{d \in D : x \preceq_d x'\}|$$

and derives the group preference from the group value function

$$x \preceq_D x' \Leftrightarrow V(x) \geq V(x')$$

Example

	Individuals						
Order	1	2	3	4	5	6	7
1	a	a	b	b	d	d	d
2	b	b	a	a	a	b	c
3	c	c	c	c	b	c	a
4	d	d	d	d	c	a	b

Since $V(a) = 2$, $V(b) = 2$, $V(c) = 0$ and $V(d) = 3$, the group preference is $d \prec a \sim b \prec c$

But d is hated by the absolute majority of the individuals

The plurality system: limitations

The plurality system can select alternatives abhorred by most individuals: it lets compact minorities prevail on disunited majorities

Moreover, the plurality system suffers from rank reversal because its value function depends on all the "winning" alternatives

Example
Let us remove b from the previous example

	Individuals						
Order	1	2	3	4	5	6	7
1	a	a	a	a	d	d	d
2	c	c	c	c	a	c	c
3	d	d	d	d	c	a	a

In this case, $V(a) = 4$, $V(c) = 0$ and $V(d) = 3$,
the group preference is $a \prec d \prec c$

By contrast, removing c would change nothing, because c never "wins"

The lexicographic method

The lexicographic method imposes a total order on the individuals

$$d_1 \prec \ldots \prec d_{|D|}$$

and applies to each pair of alternatives the first strict preference existing

$$x \preceq_D x' \ \Leftrightarrow \ \exists d \in D : x \preceq_d x' \text{ and } x \sim_{d'} x' \text{ for all } d' < d$$

In words

- the individuals are organised into a completely ordered hierarchy
- the “king” decides everything
- the “viceroy” decides all matters on which the “king” is indifferent
- ...
- the “lowest man on the totem pole” decides only matters on which everyone else is indifferent

The lexicographic method: limitation

The lexicographic method models traditional societies

(*in a rather extreme way*)

It has several advantages

- it always provides a weak order (most of the time, a total order)
- it does not suffer from rank reversal

Of course, it is not democratic, therefore

- easily unstable, unless the total order is deeply wired in culture
- inefficient, as the people on the lower levels have little incentives to contribute to the group

Does there exist a social welfare function that avoids all these problems?

The axiomatic approach will try to

- list the desired properties
- build a function that automatically satisfies them

as Von Neumann and Morgenstern for the decisions in conditions of risk