Decision Methods and Models

Master's Degree in Computer Science

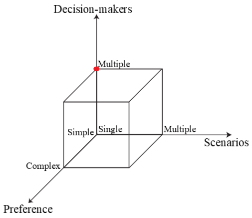
Roberto Cordone

DI - Università degli Studi di Milano

Schedule:	Thursday 16.30 - 18.30 in Aula Magna (CS department)
	Friday 12.30 - 14.30 in classroom 301
Office hours:	on appointment
E-mail:	roberto.cordone@unimi.it
Web page:	https://homes.di.unimi.it/cordone/courses/2024-mmd/2024-mmd.html
Ariel site:	https://myariel.unimi.it/course/view.php?id=4467

We assume

- multiple decision-makers: $|D| > 1$
- preference relations Π_d that are weak orders,
  possibly with a known consistent value function $u^{(d)}(f)$



We consider a special class of games with a strong property that

- guarantees the existence of Nash equilibria in pure strategies
- provides models for several well-known practical situations

Symmetric games

In a symmetric game

- all players have corresponding sets of strategies:

$$X^{(d)} = X^{(d')} \text{ for all } d, d' \in D$$

- permuting the strategies, the payoffs are correspondingly permuted

$$f^{(d)}(x^{(1)}, \dots, x^{(|D|)}) = f^{(\pi_d)}(x^{(\pi_1)}, \dots, x^{(\pi_{|D|})}) \text{ for any permutation } \pi$$

Intuitively, the payoffs depend on the strategy, not on the player

In the case of two players, a single alternative permutation exists

- $X^{(c)} = X^{(r)}$
- $f_{ij}^{(c)} = f_{ji}^{(r)}$ (*the payoffs are simply transposed*)

Example: rock paper scissors is both zero-sum and symmetric

	R	P	S
R	(0,0)	(-1,1)	(1,-1)
P	(1,-1)	(0,0)	(-1,1)
S	(-1,1)	(1,-1)	(0,0)

Symmetric games

Counterexample: odds and even is zero-sum, but not symmetric

The row player wins for equal strategies, the column for opposite ones

	Odd	Even
Odd	$(1, -1)$	$(-1, 1)$
Even	$(-1, 1)$	$(1, -1)$

The two payoffs in each strategy profile are explicitly reported, even if column player's could be derived from the row player's

A taxonomy

We will focus on symmetric games with

- two players: $D = 2$
- two strategies per player: $|X^d| = 2$ for all $d \in D$

While simple, they propose a range of interesting situations

We will present a two-level taxonomy

- 4 classes based on Nash equilibria
- 12 subclasses based on payoff orderings

An equilibrium-based taxonomy

While a general game would have $|D| \cdot \prod_{d \in D} |X^{(d)}| = 8$ payoffs,

only 4 payoffs are independent, and we assume them to be all different

$$\begin{array}{c|cc|} & 1 & 2 \\ \hline 1 & (f_{11}, f_{11}) & (f_{12}, f_{21}) \\ 2 & (f_{21}, f_{12}) & (f_{22}, f_{22}) \\ \hline \end{array}$$

Consider the first column

1. if $f_{11} > f_{21}$, consider the second column
    1. if $f_{12} > f_{22}$, a single equilibrium in $(1, 1)$ $\begin{array}{|cc|} \hline (\bar{f}_{11}, \bar{f}_{11}) & (\bar{f}_{12}, f_{21}) \\ (f_{21}, \bar{f}_{12}) & (f_{22}, f_{22}) \\ \hline \end{array}$
    2. if $f_{12} < f_{22}$, two equilibria in $(1, 1)$ and $(2, 2)$ $\begin{array}{|cc|} \hline (\bar{f}_{11}, \bar{f}_{11}) & (f_{12}, f_{21}) \\ (f_{21}, f_{12}) & (\bar{f}_{22}, \bar{f}_{22}) \\ \hline \end{array}$
2. if $f_{11} < f_{21}$, consider the second column
    1. if $f_{12} > f_{22}$, two equilibria in $(1, 2)$ and $(2, 1)$ $\begin{array}{|cc|} \hline (f_{11}, f_{11}) & (\bar{f}_{12}, \bar{f}_{21}) \\ (\bar{f}_{21}, \bar{f}_{12}) & (f_{22}, f_{22}) \\ \hline \end{array}$
    2. if $f_{12} < f_{22}$, a single equilibrium in $(2, 2)$ $\begin{array}{|cc|} \hline (f_{11}, f_{11}) & (f_{12}, \bar{f}_{21}) \\ (\bar{f}_{21}, f_{12}) & (\bar{f}_{22}, \bar{f}_{22}) \\ \hline \end{array}$

An equilibrium-based taxonomy

In summary, for all symmetric two-player two-strategy games

- there are equilibria in pure strategies
- the equilibria are one or two
- the equilibria are in symmetric strategy profiles

Games with more players or more strategies behave differently

(*see rock paper and scissors*)

A permutation-based taxonomy

We consider games with

- 2 players
- 2 strategies with conventional labels
- 4 distinct payoffs, therefore $4! = 24$ orders

Conventionally assuming $f_{11} > f_{22}$ yields 12 different games

Class	Subclass	Order	Examples
1	a	$f_{11} > f_{12} > f_{21} > f_{22}$	Ideal marriage
	b	$f_{11} > f_{12} > f_{22} > f_{21}$	
	c	$f_{11} > f_{21} > f_{12} > f_{22}$	
	d	$f_{12} > f_{11} > f_{21} > f_{22}$	
	e	$f_{12} > f_{11} > f_{22} > f_{21}$	
2	a	$f_{11} > f_{21} > f_{22} > f_{12}$	Stag hunt
	b	$f_{11} > f_{22} > f_{12} > f_{21}$	Coordination (1)
	c	$f_{11} > f_{22} > f_{21} > f_{12}$	Coordination (2)
3	a	$f_{21} > f_{11} > f_{12} > f_{22}$	Chicken's game
	b	$f_{12} > f_{21} > f_{11} > f_{22}$	Battle of the sexes (1)
	c	$f_{21} > f_{12} > f_{11} > f_{22}$	Battle of the sexes (2)
4	a	$f_{21} > f_{11} > f_{22} > f_{12}$	Prisoner's dilemma

The ideal marriage

The strategies are conventionally denoted as

1. cooperate (C)
2. not cooperate (NC)

and the payoff ordering is

$$(C, C) \prec (C, NC) \prec (NC, C) \prec (NC, NC)$$

	C	NC
C	(3, 3)	(1, 2)
NC	(2, 1)	(0, 0)

The name comes from the main features of the model:

- mutual cooperation pays more than free-riding
- free-riding pays more than being exploited
- being exploited pays more than mutual egoism

that represents an ideal case for cooperation

The ideal marriage

$$(C, C) \prec (C, NC) \prec (NC, C) \prec (NC, NC)$$

	C	NC
C	$(\bar{3}, \bar{3})$	$(\bar{1}, 2)$
NC	$(2, \bar{1})$	$(0, 0)$

Under these conditions

- noncooperation is dominated by cooperation
- there is only one Nash equilibrium in (C,C)
- the worst-case criterium leads both players to the equilibrium
- the equilibrium provides the best payoff to both players

The other games of class 1 assume $f_{12} > f_{21}$, which would mean that being exploited (C,NC) is better than free-riding (NC,C)

The names of the strategies look less appropriate

The stag hunt

The payoff ordering is

$$(C, C) \prec (NC, C) \prec (C, NC) \prec (NC, NC)$$

	C	NC
C	(3, 3)	(0, 2)
NC	(2, 0)	(1, 1)

The name comes from J. J. Rousseau's essay
Discours sur l'origine et les fondements de l'inegalité parmi les hommes:

- two hunters can cooperate and catch a stag
- one of them can defect and catch a hare (and maybe a stag), while the other gets maybe the stag
- both can defect and catch a hare

Cooperation is favoured by political structures (social contract), but noncooperation is a stable alternative

The stag hunt

$$(C, C) \prec (NC, C) \prec (C, NC) \prec (NC, NC)$$

	C	NC
C	$(\bar{3}, \bar{3})$	$(0, 2)$
NC	$(2, 0)$	$(\bar{1}, \bar{1})$

Under these conditions

- no strategy is dominated
- there are two Nash equilibria in (C,C) and (NC,NC)
- the worst-case criterium leads to the noncoopertive equilibrium
- the cooperative equilibrium provides the best payoff to both players

Once you are in a situation, it is hard to get out of it: past history rules

Pure coordination games

The payoff ordering is

$$(1,1) \prec (2,2) \prec (2,1) \prec (1,2) \quad \text{or} \quad (1,1) \prec (2,2) \prec (1,2) \prec (2,1)$$

where talking about cooperation is improper

	1	2
1	$(3,3)$	$(0,1)$
2	$(1,0)$	$(2,2)$

	1	2
1	$(3,3)$	$(1,0)$
2	$(0,1)$	$(2,2)$

- two drivers meet on a narrow road from opposite direction
- if they both drive on the right or left, they avoid each other
- if one drives on the right and the other on the left, they crash

Good results are obtained by agreeing on the same strategy, whatever it is

Pure coordination games

$(1,1) \prec (2,2) \prec (2,1) \prec (1,2)$ or $(1,1) \prec (2,2) \prec (1,2) \prec (2,1)$

	1	2
1	$(\bar{3},\bar{3})$	$(0,1)$
2	$(1,0)$	$(\bar{2},\bar{2})$

	1	2
1	$(\bar{3},\bar{3})$	$(1,0)$
2	$(0,1)$	$(\bar{2},\bar{2})$

Under these conditions

- no strategy is dominated
- there are two Nash equilibria in (1,1) and (2,2)
- the two equilibria are nearly equivalent

Once an agreement is established, it makes sense to maintain it

It is similar to the stag hunt, but more balanced

The chicken race (or hawks and doves)

The payoff ordering is

$$(NC, C) \prec (C, C) \prec (C, NC) \prec (NC, NC)$$

	C	NC
C	$(2, 2)$	$(1, 3)$
NC	$(3, 1)$	$(0, 0)$

The name derives from *Rebel without a cause*, a movie with James Dean

- two drivers drive their cars towards each other
- if both swerve together, they tie honourably
- if one swerves earlier, he is shamed and the other one is acclaimed
- if both persist, they risk their life in the crash

The largest payoff comes at a risk and cannot be obtained by both players

The chicken race (or hawks and doves)

$$(NC, C) \prec (C, C) \prec (C, NC) \prec (NC, NC)$$

	C	NC
C	$(2, 2)$	$(\bar{1}, \bar{3})$
NC	$(\bar{3}, \bar{1})$	$(0, 0)$

Under these conditions

- no strategy is dominated
- there are two Nash equilibria in (C,NC) and (NC,C)
- there is no way to know *a priori* which equilibrium will be chosen

Once you are in a situation, it is hard to get out of it: past history rules

It could depend on small asymmetries

The battle of the sexes (or anticoordination games)

The payoff ordering is

$$(2,1) \prec (1,2) \prec (1,1) \prec (2,2) \quad \text{or} \quad (1,2) \prec (2,1) \prec (1,1) \prec (2,2)$$

and talking about cooperation is once again improper

	1	2
1	$(1,1)$	$(2,3)$
2	$(3,2)$	$(0,0)$

	1	2
1	$(1,1)$	$(3,2)$
2	$(2,3)$	$(0,0)$

Two fiances want to go to a show, but cannot communicate

- the man would prefer to go to a match
- the woman would prefer to go to a ballet
- both would prefer to be together rather than alone

Example 2: a phone call is interrupted: should one wait or call again?

The battle of the sexes (or anticoordination games)

$(2,1) \prec (1,2) \prec (1,1) \prec (2,2)$ or $(1,2) \prec (2,1) \prec (1,1) \prec (2,2)$

	1	2
1	$(1,1)$	$(\bar{2},\bar{3})$
2	$(\bar{3},\bar{2})$	$(0,0)$

	1	2
1	$(1,1)$	$(\bar{3},\bar{2})$
2	$(\bar{2},\bar{3})$	$(0,0)$

Under these conditions

- no strategy is dominated
- there are two Nash equilibria in (1,2) and (2,1)
- there is no way to know *a priori* which equilibrium will be chosen

It is similar to the chicken's race, but the payoff is good for both

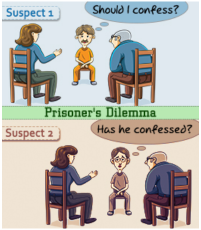
Once you are in a situation, it is hard to get out of it: past history rules

The prisoner's dilemma

The payoff ordering is

$$(NC, C) \prec (C, C) \prec (NC, NC) \prec (C, NC)$$

	C	NC
C	$(2,2)$	$(0,3)$
NC	$(3,0)$	$(1,1)$



Two gangsters arrested by the police are suspected of a major crime

- if they do not confess, the evidence is enough for a short sentence
- if one confesses, the police offers to further reduce the conviction, while sentencing the other gangster to a long conviction
- if both confess, however, they will receive an intermediate sentence

The incentive to confess cannot be gained by both players

The prisoner's dilemma

$$(NC, C) \prec (C, C) \prec (NC, NC) \prec (C, NC)$$

	C	NC
C	$(2, 2)$	$(0, 3)$
NC	$(3, 0)$	$(\bar{1}, \bar{1})$

Under these conditions

- cooperation is dominated by noncooperation
- there is only one Nash equilibrium in (NC,NC)
- the worst-case criterium leads both players to the equilibrium
- the equilibrium provides a bad payoff to both players

Cooperating would be better for both, but requires irrational trust

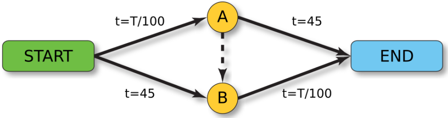
The prisoner's dilemma

The prisoner's dilemma has been applied in several different fields:

- the management of natural resources (tragedy of the commons)
- the management of traffic (Braess' paradox)

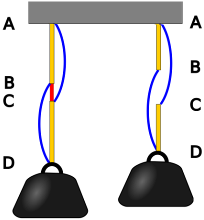
  If 4 000 drivers go from start to end

  - the shortest path uses AB and takes 80 minutes
  - removing AB, they split along two paths that take 65 minutes



- physics (the spring paradox)

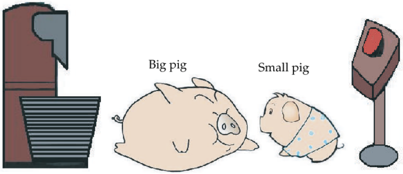
  Cutting the red rope, the yellow springs switch from series to parallel and the weight is lifted



The pigsty game (or pigeon coop game)

A large application field for game theory is ethology and evolution theory

Example



- a strong dominant pig and a weaker one share the same sty
- the two pigs can obtain food by pushing a lever
- the food is provided on the opposite side of the lever

The game is obviously nonzero-sum and nonsymmetric

The pigs have two possible strategies

- push the lever
- wait for the other pig to do it

The pigsty game (or pigeon coop game)

The resulting payoffs are

		Weak pig P	Weak pig W
Strong pig	P	$(4, 2)$	$(3, 3)$
	W	$(5, 0)$	$(1, 1)$

because

- if both pigs push the lever, the weaker one eats some food before being sent away
- if the strong pig pushes the lever, the weaker one eats more food before being sent away
- if the weaker pig pushes the lever, it eats nothing and wastes energy
- if both pigs wait, no food is provided

The pigsty game (or pigeon coop game)

		Weak pig P	Weak pig W
Strong pig	P	$(4, 2)$	$(\bar{3}, \bar{3})$
	W	$(\bar{5}, 0)$	$(1, \bar{1})$

Under these conditions

- the waiting strategy is dominating for the weaker pig
- there is a single Nash equilibrium in which
  - the strong pig pushes the lever
  - the weak pig waits and eats