Decision Methods and Models

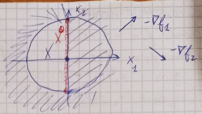
Master's Degree in Computer Science

Roberto Cordone
DI - Università degli Studi di Milano

Schedule:	Thursday 16.30 - 18.30 in Aula Magna (CS department)
	Friday 12.30 - 14.30 in classroom 301
Office hours:	on appointment
E-mail:	roberto.cordone@unimi.it
Web page:	https://homes.di.unimi.it/cordone/courses/2024-mmd/2024-mmd.html
Ariel site:	https://myariel.unimi.it/course/view.php?id=4467

Lesson 12: Paretian preference (exercises) Milano, A.A. 2024/25

$$\begin{cases} \min f_1(x) = -x_1 - x_2 \\ \min f_2(x) = -x_1 + x_2 \\ g_1(x) = x_1^2 + x_2^2 - 1 \le 0 \\ g_2(x) = x_1 \le 0 \end{cases}$$



$$\nabla f_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \nabla f_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

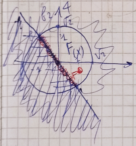
INVERSE TRANSFORMATION

$$\begin{cases} f_1 = -x_1 - x_2 \\ f_2 = -x_1 + x_2 \end{cases} \Rightarrow \begin{cases} x_1 = -x_2 - f_1 \\ x_2 = f_2 + x_1 = f_2 - x_2 - f_1 \Rightarrow 2x_2 = f_2 - f_1 \Rightarrow x_2 = \dfrac{f_2 - f_1}{2} \end{cases} \Rightarrow x_1 = -\frac{f_2}{2} + \frac{f_1}{2} - f_1 =$$

$$\Rightarrow \begin{cases} x_1 = \dfrac{-f_1 - f_2}{2} \\ x_2 = \dfrac{f_2 - f_1}{2} \end{cases} \Rightarrow \begin{cases} g_1(x(f)) = \left(\dfrac{-f_1 - f_2}{2}\right)^2 + \left(\dfrac{f_2 - f_1}{2}\right)^2 - 1 \le 0 \\ g_2(x(f)) = \dfrac{-f_1 - f_2}{2} \le 0 \end{cases} \Rightarrow \Big| \; f_1 + f_2 \ge 0 \Rightarrow f_2 \ge -f_1$$

$$\frac{f_1^2 + f_2^2 + \cancel{2f_1f_2}}{4} + \frac{f_2^2 + f_1^2 - \cancel{2f_1f_2}}{4} \le 1 \Rightarrow 2f_1^2 + 2f^2 \le 4 \Rightarrow f_1^2 + f_2^2 \le 2$$

$$f_2 = -f_1 \Rightarrow f_1^2 + (-f_1)^2 = 2 \Rightarrow$$
$$\Rightarrow 2f_1^2 = 2 \Rightarrow f_1 = \pm 1$$
$$f_2 = \mp 1$$



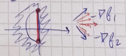
$$\begin{cases} f_1 = \alpha \\ f_2 = -\alpha \end{cases} \text{ with } \alpha \in [-1, 1]$$

$$x(f(\alpha)) = \begin{cases} x_1 = \dfrac{-f_1 - f_2}{2} = \dfrac{-\alpha + \alpha}{2} = 0 \\ x_2 = \dfrac{-\alpha - \alpha}{2} = -2\alpha \end{cases} \text{ with } \alpha \in [-1, 1)$$

$$\begin{cases} f_2 = \alpha \\ f_1 = -\alpha \end{cases} \Rightarrow \begin{cases} x_1 = \dfrac{\alpha - \alpha}{2} = 0 \\ x_2 = \dfrac{\alpha + \alpha}{2} = 2\alpha \end{cases} \text{ with } \alpha \in [-1, 1]$$

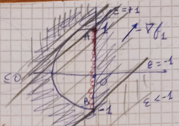
WEIGHTED SUM METHOD

$$\min z_w(x) = w(-x_1 - x_2) + (1 - w)(-x_1 + x_2) =$$
$$= \cancel{-wx_1} - wx_2 - x_1 + x_2 + \cancel{wx_1} - wx_2$$
$$= -x_1 + (1 - 2w)x_2$$



ε - CONSTRAINT METHOD

$$\begin{cases} \min\ f_1 = -x_1 - x_2 \\ \cancel{\min\ f_2 = -x_1 + x_2} \\ g_1(x) = x_1^2 + x_2^2 - 1 \le 0 \\ g_2(x) = x_1 \le 0 \end{cases}$$



$-x_1 + x_2 \le \varepsilon \Rightarrow x_2 \le x_1 + \varepsilon$

For $\varepsilon \geq +1$: $x^\circ = (0,1) = A$

$1 > \varepsilon \geq -1$: $x^\circ \in$ segment from $(0,-1)$ to $(0,1) = \overline{AB}$

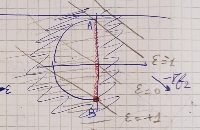
$\varepsilon < -1$: NO SOLUTION

$\Big\}\ X^{\varepsilon c} = \overline{AB}$

$\min\ f_2 = -x_1 + x_2$

$x_1^2 + x_2^2 - 1 \le 0$

$x_1 \le 0$



$-x_1 - x_2 \le \varepsilon \Rightarrow x_2 \geq -x_1 - \varepsilon$

For $\varepsilon \geq 1$: $x^\circ = B = (0,-1)$

For $-1 > \varepsilon \geq -1$: $x = \overline{AB}$

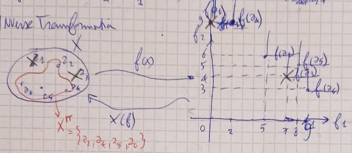
For $\varepsilon < -1$: NO SOLUTION

$\Big\}\ X^{\varepsilon c} = \overline{AB}$

Benefits	a_1	a_2	a_3	a_4	a_5	a_6
f_1	0	5	7	2	8	9
f_2	9	6	4	9	5	3
z_w	$w\cdot 0+(1-w)9$					

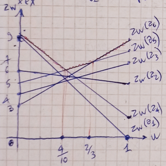
Definition: $a_4 \prec a_1 \quad a_5 \prec a_3 \Rightarrow X^o = \{a_2, a_4, a_5, a_6\}$

Inverse Transformation



WEIGHTED SUM METHOD

$$\max_{x\in X} z_w = w f_1 + (1-w) f_2 .$$



$$z_w(a_1) = w\cdot 0 + (1-w)\cdot 9 = 9 - 9w$$
$$z_w(a_2) = w\cdot 5 + (1-w)\cdot 6 = -w + 6$$
$$z_w(a_3) = w\cdot 7 + (1-w)\cdot 4 = 3w + 4$$
$$z_w(a_4) = w\cdot 2 + (1-w)\cdot 9 = -7w + 9$$
$$z_w(a_5) = w\cdot 8 + (1-w)\cdot 5 = 3w + 5$$
$$z_w(a_6) = w\cdot 9 + (1-w)\cdot 3 = 6w + 3$$

$$X^{WS} = \{a_4, a_5, a_6\} \subset X^o$$

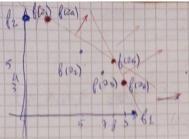
$z_w(a_4) = z_w(a_5) \Rightarrow -7w + 9 = 3w + 5 \Rightarrow 10w = 4 \Rightarrow w = 4/10$

$z_w(a_5) = z_w(a_6) \Rightarrow 3w + 5 = 6w + 3 \Rightarrow 3w = 2 \Rightarrow w = 2/3$

$\mathrm{Supp}(a_4) = \left(0, \frac{4}{10}\right]$ $\quad \mathrm{Supp}(a_2) = \emptyset$

$\mathrm{Supp}(a_5) = \left[\frac{4}{10}, \frac{2}{3}\right]$

$\mathrm{Supp}(a_6) = \left[\frac{2}{3}, 1\right)$



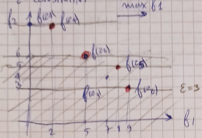
$\max f_1 \rightarrow$

$\max f_2 \uparrow$

$\max z_w = w f_1 + (1-w) f_2$

$\Rightarrow X^{WS} = \{a_4, a_5, a_6\}$

ε- CONSTRAINT



$$\begin{cases} \max f_1 \\ x \in X \\ f_2 \geq \varepsilon \end{cases}$$

For $\varepsilon \leq 3 \Rightarrow x^\circ = a_6$

For $3 < \varepsilon \leq 4 \Rightarrow$ } $x^\circ = a_5$

For $4 < \varepsilon \leq 5 \Rightarrow$ }

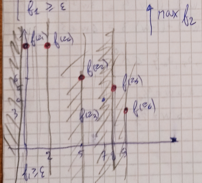
For $5 < \varepsilon \leq 6 \Rightarrow x^\circ = a_2$

For $6 < \varepsilon \leq 9 \Rightarrow x^\circ = a_4$

For $\varepsilon > 9 \Rightarrow$ NO SOLUTION

$X^{\varepsilon c_1} = \{a_2, a_4, a_5, a_6\}$

$$\begin{cases} \max f_2 \\ x \in X \\ f_1 \geq \varepsilon \end{cases}$$



For $\varepsilon \leq 0 \Rightarrow X^\circ = a_1$, OR a_4

For $0 < \varepsilon \leq 2 \Rightarrow x^\circ = a_4$

For $2 < \varepsilon \leq 5 \Rightarrow x^\circ = a_2$

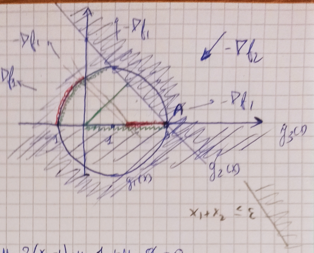
For $5 < \varepsilon \leq 8 \Rightarrow x^\circ = a_5$

For $\varepsilon \leq 9 \Rightarrow x^\circ = a_6$

For $\varepsilon > 9 \Rightarrow$ NO SOLUTION

$X^{\varepsilon c_2} = \{a_1, a_2, a_4, a_5, a_6\} \supset X$

$$\begin{cases} \min\ f_1(x) = -x_1^2 - x_2^2 \\ \min\ f_2(x) = x_1 + x_2 \\ g_1(x) = (x_1-1)^2 + x_2^2 - 4 \le 0 \\ g_2(x) = x_1 + x_2 - 3 \le 0 \\ g_3(x) = -x_2 \le 0. \end{cases}$$



$$\nabla f_1 = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix} \quad \nabla f_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

KKT:
$$\begin{cases} w\cdot(-2x_1) + (1-w)\cdot 1 + \mu_1 2(x_1-1) + \mu_2 \cdot 1 + \mu_3 \cdot \emptyset = 0 \\ w\ (-2x_2) + (1-w)\, 1 + \mu_1 \cdot 2x_2 + \mu_2 1 + \mu_3(-1) = 0 \\ \mu_1 \left[(x_1-1)^2 + x_2^2 - 4 \right] = 0 \\ \mu_2 (x_1 + x_2 - 3) = 0 \\ \mu_3 \cdot x_2 = 0 \\ \mu_j \ge 0 \\ g_j(x) \le 0 \\ 0 \le w \le 1 \end{cases}$$

$A = (3,0)$ is NON REGULAR

8 problems
$$\begin{cases} \mu_1 = 0 \text{ or } \mu_1 > 0 \\ \mu_2 = 0 \text{ or } \mu_2 > 0 \\ \mu_3 = 0 \text{ or } \mu_3 > 0 \end{cases}$$

P_{000}: $\mu_1 = \mu_2 = \mu_3 = 0$: $\begin{cases} -2wx_1 + 1 - w = 0 \\ -2wx_2 + 1 - w = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1-w}{2w} \\ x_2 = \frac{1-w}{2w} \end{cases}$

P_{001}: $\mu_1 = \mu_2 = 0$, $\mu_3 > 0 \Rightarrow x_2 = 0$ $\begin{cases} -2wx_1 + 1 - w = 0 \\ -\cancel{2wx_2} + 1 - w - \mu_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1-w}{2w},\ x_2 = 0 \\ \mu_3 = 1 - w \ge 0 \end{cases}$