Decision Methods and Models

Master's Degree in Computer Science

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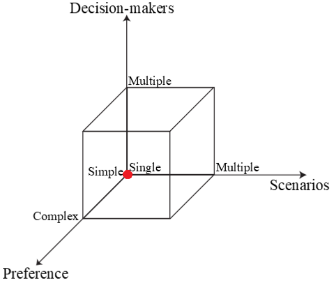
Structured models of preference

We assume

- a preference relation Π with a consistent utility function $u(f)$
- a certain environment: $|\Omega| = 1 \Rightarrow f(x, \bar{\omega})$ reduces to $f(x)$
- a single decision-maker: $|D| = 1 \Rightarrow \Pi_d$ reduces to Π



We known the preference Π, not the utility function $u(f)$

The general process to find it is complex and error-prone

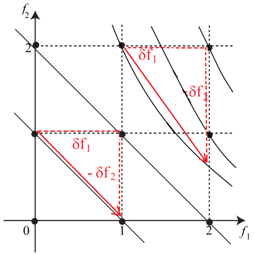
For additive functions, it is much simpler, but is $u(f)$ additive?

- for $p \geq 3$, mutual preferential independence $\Leftrightarrow$ additivity
- for $p = 2$, mutual preferential independence $\Leftarrow$ additivity

The problem lies in how the indifference curves behave in F

Marginal rate of substitution (*MRS*)

The behaviour of a curve is described by the relation between f_1 and f_2:
from f, vary f_1 and update f_2 so as to remain on the indifference curve



Marginal rate of substitution (*MRS*) of f_1 with f_2 in f

$$\lambda_{12}(f) = \lim_{\delta f_1 \to 0} - \frac{\delta f_2(f, \delta f_1)}{\delta f_1}$$

with $\delta f_2(f, \delta f_1)$ such that $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \sim \begin{bmatrix} f_1 + \delta f_1 \\ f_2 + \delta f_2(f, \delta f_1) \end{bmatrix}$

- the value depends on f, not on δf_1, thanks to the limit operation
- the minus sign is used to obtain positive rates on decreasing curves (a frequent case: e.g., when both indicators are costs or benefits)

Equivalent expressions of the *MRS* (1)

Let the indifference curve be represented as a regular parametric line:

$$\begin{cases} f_1 = f_1(\alpha) \\ f_2 = f_2(\alpha) \end{cases}$$

with two functions of parameter α (continuous up to the first derivative)

The *MRS* can be expressed as

$$\lambda_{12}(f) = \lim_{\delta\alpha\to 0} -\frac{(f_2(\alpha+\delta\alpha) - f_2(\alpha))}{(f_1(\alpha+\delta\alpha) - f_1(\alpha))} = \frac{\lim_{\delta\alpha\to 0} -\frac{(f_2(\alpha+\delta\alpha) - f_2(\alpha))}{\delta\alpha}}{\lim_{\delta\alpha\to 0} \frac{(f_1(\alpha+\delta\alpha) - f_1(\alpha))}{\delta\alpha}} = -\frac{\frac{df_2}{d\alpha}}{\frac{df_1}{d\alpha}}$$

This form is useful to prove reciprocity: $\lambda_{12}(f) = \frac{1}{\lambda_{21}(f)}$

Example (Cobb-Douglas):

$$\begin{cases} f_1 = \frac{1}{\sqrt{\alpha}} \\ f_2 = \sqrt[3]{\alpha} \end{cases} \Rightarrow \lambda_{12}(f) = -\frac{\frac{1}{3}\alpha^{-2/3}}{-\frac{1}{2}\alpha^{-3/2}} = \frac{2}{3}\frac{f_2}{f_1}$$

(multiply both numerator and denominator by α)

Equivalent expressions of the *MRS* (2)

A second expression shows the relation between the *MRS* and $u(f)$

By changing α, we move on the indifference curve, but $u(f(\alpha)) = c$

$$\frac{du(f(\alpha))}{d\alpha} = 0 \Rightarrow \frac{\partial u}{\partial f_1}\frac{df_1}{d\alpha} + \frac{\partial u}{\partial f_2}\frac{df_2}{d\alpha} = 0 \Rightarrow -\frac{\frac{df_2}{d\alpha}}{\frac{df_1}{d\alpha}} = \lambda_{12}(f) = \frac{\frac{\partial u}{\partial f_1}}{\frac{\partial u}{\partial f_2}}$$

The *MRS* measures how much $u(f)$ depends on f_1
with respect to how much it depends on f_2

Example (Cobb-Douglas):

$$u(f) = f_1^2 f_2^3 \Rightarrow \lambda_{12}(f) = \frac{2f_1 f_2^3}{3f_1^2 f_2^2} = \frac{2}{3}\frac{f_2}{f_1}$$

Equivalent expressions of the *MRS* (3)

The third expression shows the geometric relation between the *MRS* and the shape of indifference curves (if invertible!)

$$\begin{cases} f_1 = f_1(\alpha) \\ f_2 = f_2(\alpha) \Leftrightarrow \alpha = \alpha(f_2) \end{cases} \Rightarrow f_1 = f_1(\alpha(f_2))$$

which implies that

$$\lambda_{12}(f) = -\frac{df_2}{d\alpha}\frac{d\alpha}{df_1} = -\frac{df_2}{df_1}$$

The *MRS* is the negative slope of the tangent to the indifference curve

Example (Cobb-Douglas):

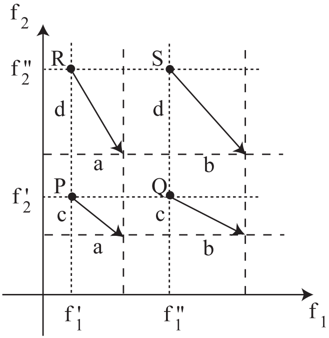
$$u(f) = f_1^2 f_2^3 = c \Rightarrow f_2 = \sqrt[3]{\frac{c}{f_1^2}}$$

implies that

$$\lambda_{12}(f) = -\sqrt[3]{c}\left(-\frac{2}{3}\right) f_1^{-5/3} = \frac{2}{3}\frac{f_2}{f_1}$$

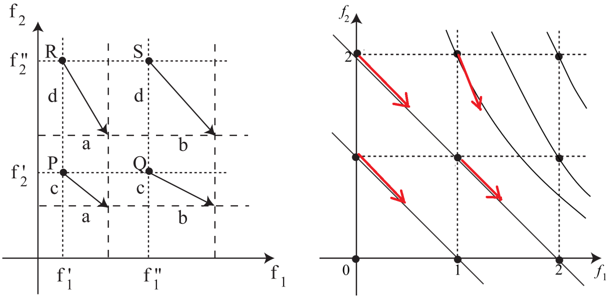
Some indifference maps enjoy the corresponding trade-off condition

$$\lambda_{12}(f_1', f_2') \cdot \lambda_{12}(f_1'', f_2'') = \lambda_{12}(f_1', f_2'') \cdot \lambda_{12}(f_1'', f_2')$$



This is a global property: it relates far away impacts

Corresponding trade-off condition



The corresponding trade-off condition is easier to interpret if rewritten as

$$\lambda_{12}(f_1', f_2')/\lambda_{12}(f_1', f_2'') = \lambda_{12}(f_1'', f_2')/\lambda_{12}(f_1'', f_2'')$$

$$\lambda_{12}(f_1', f_2')/\lambda_{12}(f_1'', f_2') = \lambda_{12}(f_1', f_2'')/\lambda_{12}(f_1'', f_2'')$$

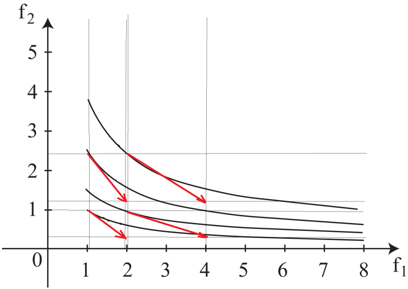
The ratio of λ between points with the same abscissa (ordinate) does not depend on the abscissa (ordinate)

In other words, even is the *MRS* is nonuniform,
it changes by the same factor moving between the same coordinates

The example on the right violates it in $(0,1)$, $(1,1)$, $(0,2)$, $(1,2)$

Corresponding trade-off condition: positive example

On the contrary, the condition holds for the Cobb-Douglas example



In fact

$$\frac{2}{3}\frac{f_2'}{f_1'} \cdot \frac{2}{3}\frac{f_2''}{f_1''} = \frac{2}{3}\frac{f_2'}{f_1''} \cdot \frac{2}{3}\frac{f_2''}{f_1'}$$

Corresponding trade-off condition and additivity

Theorem

A preference relation Π admits an additive utility function $u(f)$ if and only if it enjoys both

1. mutual preferential independence
2. the corresponding trade-off condition

The "only-if" part is easy to prove

We needed the "if" part to close the gap in order to assume additivity:

- when $p \geq 3$, check mutual preferential independence for $p - 1$ indicator pairs
- when $p = 2$, check
  - independence for the single indicators
  - the corresponding trade-off condition

Building an additive utility function

$$u(f) = \sum_{l=1}^{p} u_l(f_l)$$

This expression assumes the same measure unit for all u_l functions

In practice, experts in different fields use different units. Therefore:

1. adopt normalised utilities: pure numbers $\tilde{u}_l(f_l)$ instead of $u_l(f_l)$
2. introduce weights w_l to combine them

$$u(f) = \sum_{l=1}^{p} w_l \tilde{u}_l(f_l)$$

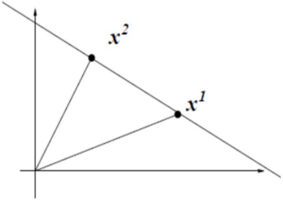
Intuitively, we are splitting the task in two

1. "rescale" indicators into utilities, removing all nonlinearities
2. "combine" heterogeneous utilities into a single one

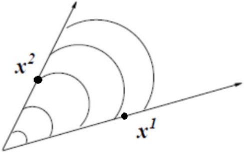
The normalised utility expression is more than a linear combination

It is a convex combination

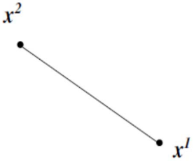
- conic: all coefficients are nonnegative ($w_l \geq 0$ for all $l \in P$)
- affine: the coefficients have unitary sum ($\sum_{l=1}^{p} w_l = 1$)



Affine combination



Conical combination



Convex combination

Building a normalised utility: the bisection method

Build the normalised utility $\tilde{u}_C(C)$ for the daily calorie intake C

Given $F_C = [0, 10000]$, interview an expert about a specific individual

1. ask the expert for the worst values in F_C (for example, $C_1^\dagger \leq 1000$ and $C_2^\dagger \geq 6000$) and set $\tilde{u}_C(C) = 0$ for such values
2. ask the expert for the best values in F_C (say, $2200 \leq C^\circ \leq 2600$) and set $\tilde{u}_T(22) = 1$ for such values
3. ask the expert for values of exactly intermediate utility between $T^\dagger$ and T° (for example, $C = 1800$ and 3000) and set $\tilde{u}_C(C) = 1/2$
4. go on, asking for values of intermediate utility between the fixed ones and set $\tilde{u}_T$ accordingly
5. guess an interpolating function

Costs and benefits proportional to the indicator f_l are easy to normalise

$$\tilde{u}_l(f_l) = \frac{f_l - \min_{x \in X} f_l(x)}{\max_{x \in X} f_l(x) - \min_{x \in X} f_l(x)}$$

What if $\min_x f_l(x)$ or $\max_x f_l(x)$ are unknown or hard to compute?

Computing the weights

As in the general case, find $p-1$ independent pairs of indifferent impacts

- the equations are linear in w

$$f \sim f' \Leftrightarrow \tilde{u}(f) = \tilde{u}(f') \Leftrightarrow \sum_{l=1}^{p} \tilde{u}_l(f_l)\, w_l = \sum_{l=1}^{p} \tilde{u}_l(f'_l)\, w_l$$

- the normalisation condition imposes convexity

$$\sum_{l=1}^{p} w_l = 1$$

The process works correctly in the ideal case

- Problem: if indifference is imprecise, $(p-1)$ pairs give wrong weights
- Solution: build a complete pairwise comparison of all indicators and analyse its consistency

Pairwise comparison matrix

Select a pair of indifferent impacts (f, f') with

- different values for f_l and f_m
- identical values for all other indicators: $f_n = f'_n$ for all $n \in P \setminus \{l, m\}$

The equation reduces to

$$w_l \tilde{u}_l(f_l) + w_m \tilde{u}_m(f_m) = w_l \tilde{u}_l(f'_l) + w_m \tilde{u}_m(f'_m)$$

that simply becomes

$$\tilde{\lambda}_{lm} = \frac{w_l}{w_m} = -\frac{\tilde{u}_m(f'_m) - \tilde{u}_m(f_m)}{\tilde{u}_l(f'_l) - \tilde{u}_l(f_l)}$$

The pairwise comparison matrix contains all the weight ratios

$$\tilde{\Lambda} = \left\{ \frac{w_l}{w_m} \right\}$$

expressing the relative weights between the single normalised utilities, that is, indicators (once nonlinearities and units of measures are removed)

Properties of the pairwise comparison matrix

A correct pairwise comparison matrix $\tilde{\Lambda}$ enjoys the following properties:

1. positivity: $\tilde{\lambda}_{lm} > 0$ for all $l, m \in P$
2. reciprocity: $\tilde{\lambda}_{lm} = \dfrac{1}{\tilde{\lambda}_{ml}}$ for all $l, m \in P$
3. consistency: $\tilde{\lambda}_{ln} = \tilde{\lambda}_{lm}\,\tilde{\lambda}_{mn}$ for all $l, m, n \in P$

Check these properties on $\tilde{\Lambda}$ to be sure that $u(f)$ makes sense

We shall discuss what to do when they are not satisfied