Decision Methods and Models

Master's Degree in Computer Science

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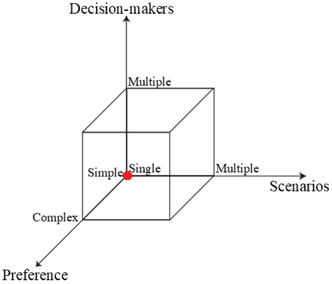
Structured models of preference

We assume

- a simple preference relation Π *What does "simple" mean?*
- a certain environment: $|\Omega| = 1 \Rightarrow f(x, \bar{\omega})$ reduces to $f(x)$
- a single decision-maker: $|D| = 1 \Rightarrow \Pi_d$ reduces to Π



and discuss

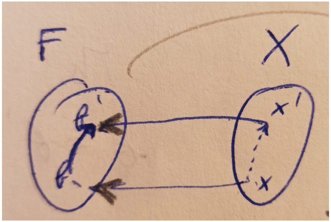
- the general advantages of such simple models
- their relation with classical optimisation problems
- some specific models of this family

(*the next lessons will introduce a more general model*)

Dominance relation

The preference relation between impacts ($\Pi \subseteq F \times F$) projects onto an induced relation between solutions

Definition (for all $x, x' \in X$): $x \preceq x' \Leftrightarrow f(x) \preceq f(x')$



This implies a partition of the feasible region into

- dominated solutions: $x \in X$ such that $\exists x' \in X : x' \prec x$
- nondominated solutions X°: the other ones

Dominance and decision problems

Reflexivity looks very natural in a preference relation

When solving a decision problem, it is also rather natural to:

- exclude dominated solutions, that is choose $x^\circ \in X^\circ$
- choose an arbitrary solution from a set of mutually indifferent ones

but this conflicts with some possible situations:

- all solutions in a strict dominance circuit would be removed
- two solutions might be indifferent with respect to a third one, but incomparable with each other

Transitivity solves both problems

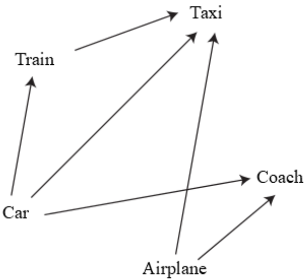
$\Rightarrow$ preorders are strong candidates to be preference relations

But decisions refer to dominance, not preference

Decision-making on preorders

Theorems:

- if preference Π is a preorder, the induced dominance is a preorder
- if preference Π is a preorder and X is finite and nonempty, nondominated solutions exist ($X^\circ \neq \emptyset$)



Strict preference graph

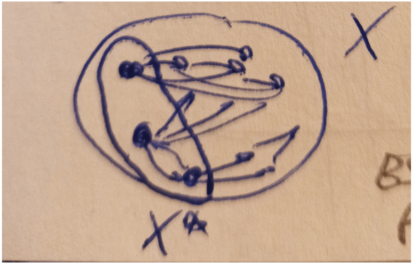
A finite X avoids infinite dominance chains (every solution is dominated)

More complex conditions allow an extension to infinite sets

Decision-making on preorders

Theorem:

- if preference Π is a preorder and X° is nonempty, the nondominated solutions partition into disjoint components
  - they are mutually indifferent within each component
  - they are mutually incomparable between different components



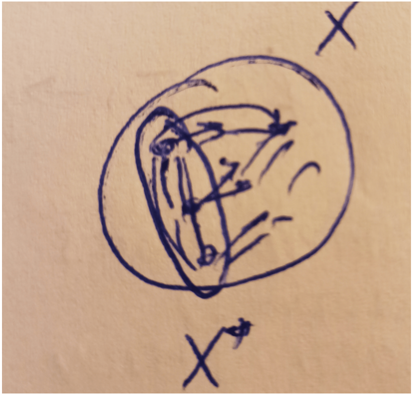
$\Rightarrow$ If there is only one component, the problem is solved

That would require completeness (no incomparability)

Decision-making on weak orders

Theorems:

- if preference Π is a weak order,
  the induced dominance is a weak order
- if preference Π is a weak order and X is finite and nonempty,
  nondominated solutions exist and are all mutually indifferent



Once again, an extension to infinite sets is possible

What about partial or total orders?

Partial and total orders are not preserved!

- antisymmetry in Π can be lost during the projection

$$x \neq x' \not\Rightarrow f(x) \neq f(x')$$

ADD A NICE PICTURE

But this is not actually a problem for decision making

Decision-making and classical optimisation

A value function on F ($v : F \to \mathbb{R}$), called utility function in economics, is any function associating real values to the elements of F

A value function is consistent with a preference relation Π when

$$f \preceq f' \Leftrightarrow v(f) \geq v(f') \text{ for each } f, f' \in F$$

that is

$$\Pi = \{(f, f') \in F \times F : v(f) \geq v(f')\}$$

This offers a compact way to represent preference relations

That is also good for computation

$$\max v(f(x))$$
$$x \in X$$

if we have analytic expressions for X and $v(f(\cdot))$ and a solving algorithm

Value functions are not univocal (infinite equivalent ones always exist)

Relation between value functions and weak orders

Theorem:

If Π admits a consistent value function $v(f)$, then Π is a weak order

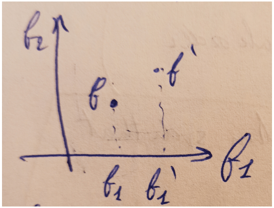
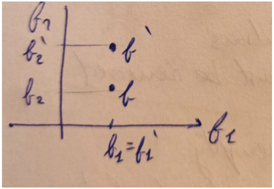
In practice, we start from a preference relation, not from a value function

The converse would be more useful

The converse is not always true

The main counterexample is lexicographic preference

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \preceq \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} \Leftrightarrow f_1 < f_1' \text{ or } (f_1 = f_1' \text{ and } f_2 < f_2')$$

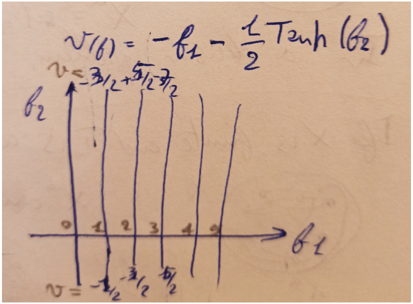



Informally, the ∞^2 impacts are all different,

$\Rightarrow$ they cannot be mapped on ∞ real values remaining all different

Lexicographic preference

By contrast, they can if one component has infinite, but discrete, values



$$v(f) = -f_1 - \frac{1}{2}\tanh f_2$$

mapping all values of f_2 for $f_1 = k$ on interval $(-k - 1/2, -k + 1/2)$

Weak order preference models: scalar impact

When the impact is one-dimensional, it is often easy (though not always) to turn it into a value function

- if the impact is a benefit: just set $v(f) = f$
  Examples: profit, quality, ...
- if the impact is a cost: just set $v(f) = -f$
  Examples: monetary cost, time, pollution, ...
- if the impact has a target value $\bar{f}$: just set $v(f) = -\mathrm{dist}(f, \bar{f})$
  Examples: a goal to reach, an ideal impact, ...
- ...

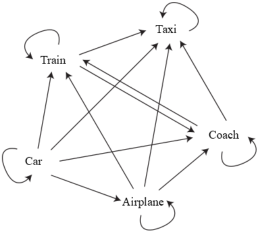
(*but what if the problem is multidimensional or hard to model?*)

Weak order preference models: the Borda count

In the finite case, every weak order admits a value function (Borda count)

$$B(f) = |\{f' \in F : f \preceq f'\}|$$

Example:



X	$v(f(x))$
Airplane	4
Car	5
Coach	3
Taxi	1
Train	3

This is not very useful to solve the problem: better to apply domination
(*but we will use it for group decisions*)

(*but what if the problem is not finite?*)

Weak order preference models: lexicographic order

If the indicators are all costs (or benefits) and are sorted by importance ($P = (\pi_1, \pi_2, \ldots, \pi_p)$), the preference relation Π is a total order

$$f \preceq f' \Leftrightarrow f_{\pi_1} < f'_{\pi_1} \text{ or } \left\{ \begin{array}{l} f_{\pi_1} = f'_{\pi_1} \\ f_{\pi_2} < f'_{\pi_2} \end{array} \right\} \text{ or } \ldots \text{ or } \left\{ \begin{array}{c} f_{\pi_1} = f'_{\pi_1} \\ f_{\pi_2} = f'_{\pi_2} \\ \ldots \\ f_{\pi_p} \leq f'_{\pi_p} \end{array} \right\}$$

Examples:

- first find a trip with minimum cost, then with minimum time
- first maximise the serviced demand, then minimise the costs

It does not admit value functions, but it can be solved as follows:

1) find the whole set $X^\circ_{\pi_1}$ of optimal solutions for $\min_{x \in X} f_{\pi_1}(x)$
2) find the whole set $X^\circ_{\pi_2}$ of optimal solutions for $\min_{x \in X^\circ_{\pi_1}} f_{\pi_2}(x)$

.) ...

p) find a single optimal solution $x^\circ_{\pi_p}$ for $\min_{x \in X^\circ_{\pi_{p-1}}} f_{\pi_p}(x)$

(*but what if the indicators are not absolutely hierarchical?*)

Utopia is an inexistent ideal place (Thomas More, 1516)

- *ou-tòpos* = no place
- *eu-tòpos* = good place

The utopia point model of preference

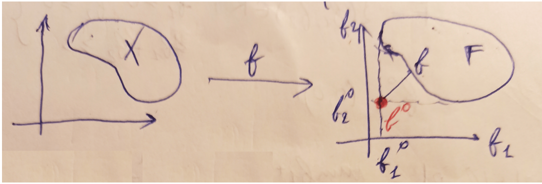
1. identifies an ideal impact f° independently optimising each indicator

$$f_l^\circ = \min_{x \in X} f_l(x)$$

   and combining the optimal values in a vector: $f^\circ = \left[f_1^\circ \dots f_p^\circ \right]^T$

2. finding a solution with impact having minimum "distance" from f°

$$\min_{x \in X} \mathrm{dist}(f(x), f^\circ)$$



That seems to make sense

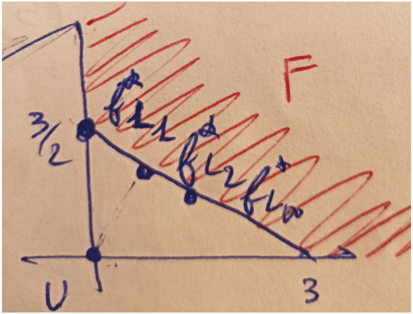
Weak order preference models: utopia point

Different definitions of distance imply different results
and the choice is arbitrary

For the sake of simplicity,
assume that $f(x) = x$ $(F = X)$

Then, $f^\circ = (0,0)$ and we solve

$$\begin{aligned} \min \mathrm{dist}(f(x),(0,0)) & & \\ x_1 + 2x_2 &\geq& 3 \\ x_1 &\geq& 0 \\ x_2 &\geq& 0 \end{aligned}$$



The optimal solution is

- $(0, 3/2)$ for distance $L_1 = |f_1| + |f_2|$
- $(3/5, 6/5)$ for distance $L_2 = \sqrt{f_1^2 + f_2^2}$
- $(1,1)$ for distance $L_\infty = \max(f_1, f_2)$

If the indicators are heterogeneous, the units of measure have an influence and arbitrary conversion coefficients are required