Decision Methods and Models

Master's Degree in Computer Science

Roberto Cordone

DI - Università degli Studi di Milano

Schedule:	Thursday 16.30 - 18.30 in Aula Magna (CS department)
	Friday 12.30 - 14.30 in classroom 301
Office hours:	on appointment
E-mail:	roberto.cordone@unimi.it
Web page:	https://homes.di.unimi.it/cordone/courses/2024-mmd/2024-mmd.html
Ariel site:	https://myariel.unimi.it/course/view.php?id=4467

Decision models: terminology

System is the portion of the world affected by the decision

- a system should allow different configurations that combine
    - an alternative or solution, describing all its controllable aspects
    - a scenario or outcome, describing all its uncontrollable aspects
- each configuration is associated to an impact, that describes all aspects which are relevant for the decision
- decision-maker or stakeholder is everybody who contributes to the choice of an alternative
- the preference describes the relative satisfaction between impacts

A decision problem requires to choose an alternative

- so as to move the system into a configuration
- such that the decision-makers prefer its associated impact to those of the other configurations
- keeping into account that the actual configuration depends on the alternative, but also on the scenario

Example: the thermostat regulation

We have discussed examples of how to identify these elements

In the problem of tuning the temperature inside a classroom

- system is the classroom
- alternative is the position of the thermostat knob
  or the degree of window openings
- scenario is the external temperature
  and the exposition of the classroom to the sun
- impact is the internal temperature and humidity of the classroom
- decision-makers are the people dwelling in the classroom
  (all of them, or just the teacher?)

Lessons derived from the case studies

When analysing the two case studies, we have remarked that:

- the alternatives are generated combining elements of alternative and there is always an alternative zero (leaving the system as it is)
- the scenarios are generated combining elements of scenario
- the impacts are generated combining indicators and indicators are organised into a hierarchy
- each configuration generates an impact through descriptive models
- decision-makers should never ignore stakeholders (people who are affected by the decision and can react to it)
- preferences are expressed on impacts (not on configurations)

Formalisation

We will introduce

- formal definitions for the concepts informally discussed above
- conceptual problems related to these definitions
- properties of these formal objects that are desirable, but limiting: effectiveness and usefulness conflict with realism and precision

We always work with compromises

A decision problem is defined by the 6-uple

$$P = (X, \Omega, F, f, D, \Pi)$$

Feasible region

A decision problem is defined by a 6-uple

$$P = (X, \Omega, F, f, D, \Pi)$$

The feasible region X is the set of all alternatives

We will assume throughout the course that

$$X \subseteq \mathbb{R}^n \quad \Leftrightarrow \quad x = [x_1 \ldots x_n]^T$$

with a finite number n of elements of alternative or decision variables

$$x_i \in \mathbb{R} \text{ for all } i \in \{1, \ldots, n\}$$

Examples of decision variables

Real numbers can model several different situations

- binary: something is done ($x_i = 1$) or not ($x_i = 0$)

$$x_{\text{train}}, x_{\text{tram}}, x_{\text{tram-train}} \in \{0,1\} \text{ with } x_{\text{train}} + x_{\text{tram}} + x_{\text{tram-train}} = 1$$

- enumerative ($x_i \in \mathbb{N}$): integers represent cases

$$x_{\text{tech}} = 1\ (train), x_{\text{tech}} = 2\ (tram), x_{\text{tech}} = 3\ (tram - train)$$

usually for representation, not for computation!

- qualitative ($x_i \in \mathbb{N}$): integers represent grades on a qualitative scale

$$x_q = 1\ (bad), x_q = 2\ (average), x_q = 3\ (good)$$

- quantitative ($x_i \in \mathbb{N}$ or $x_i \in \mathbb{R}$): physical quantities
- functions within families: the $x_i \in \mathbb{R}$ are coefficients

$$\text{polynomial } p(t) = x_0 t^n + x_1 t^{n-1} + x_2 t^{n-2} + \ldots + x_{n-1}$$

We do not treat infinite dimensional spaces (trajectories, optimal control)

A classification of feasible regions

Correspondingly, several kinds of feasible regions X exist

- continuous
- discrete
  - infinite
  - finite
    - combinatorial (typically $|X| \in \Omega(d^n)$ for some $d > 1$)
    - "finite" (typically $|X| \in O(1)$)

Focusing on a specific kind allows algorithms that are

- more effective
- less general

Sample space

A decision problem is defined by a 6-uple

$$P = (X, \Omega, F, f, D, \Pi)$$

The sample space Ω is the set of all scenarios or outcomes

(*of a random experiment, not of the decision!*)

They also describe modelling errors (disturbances)

We will assume throughout the course that

$$\Omega \subseteq \mathbb{R}^r \quad \Leftrightarrow \quad \omega = [\omega_1 \dots \omega_r]^T$$

with a finite number r of elements of scenario or exogenous variables

$$\omega_k \in \mathbb{R} \text{ for all } k \in \{1, \dots, r\}$$

Examples and classification are analogous to the decision variables

Indicator space

A decision problem is defined by a 6-uple

$$P = (X, \Omega, F, f, D, \Pi)$$

The indicator space F is the set of all impacts

We will assume throughout the course that

$$F \subseteq \mathbb{R}^p \quad \Leftrightarrow \quad f = [f_1 \dots f_p] \text{ with } f_l \in \mathbb{R}$$

with a finite number p of

- indicators
- criteria
- attributes
- objectives (only if they must be minimised or maximised)

Impact function

A decision problem is defined by a 6-uple

$$P = (X, \Omega, F, f, D, \Pi)$$

The impact function $f : X \times \Omega \to F$ is a vectorial function that associates each configuration (x, ω) to an impact $f(x, \omega)$

Trivial example:

- buy quantities x_i of products (n decision variables)
- pay costs ω_i ($r = n$ exogenous variables)
- the total (monodimensional: $p = 1$) cost is $f(x, \omega) = \sum_{i=1}^{n} \omega_i x_i$

Representations of the impact function

The impact function can be

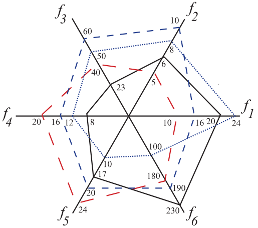
- a mathematical expression that describes a computation
- a simulation or integration software
- an empirically generated graph or table

Two classical representations for finite problems are

1. the evaluation matrix

$f(x,\omega)$	f_1	f_2	f_3	f_4	f_5	f_6
(x',ω')	10	5	40	20	24	180
(x',ω'')	16	10	60	16	20	190
(x'',ω')	20	6	23	8	17	230
(x'',ω'')	24	8	50	12	10	100

2. the radar chart



With benefits (costs), larger (smaller) polygones are better

Decision-makers

A decision problem is defined by a 6-uple

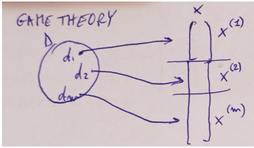
$$P = (X, \Omega, F, f, D, \Pi)$$

Set D is a finite set that includes all stakeholders

They (directly or indirectly) set the value of the decision variables x

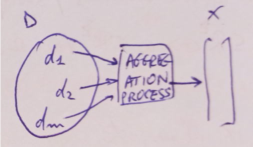
1. independently

$$X = X^{(1)} \times \ldots \times X^{(d)}$$



2. cooperatively

$$X = X^{(1)} \cap \ldots \cap X^{(d)}$$



Preference function

A decision problem is defined by a 6-uple

$$P = (X, \Omega, F, f, D, \Pi)$$

The preference function $\Pi : D \to 2^{F \times F}$
associates each decision-maker $d \in D$ to a subset of impact pairs
that represents a binary relation $\Pi(d)$ on F

What subset is this?

Subset $\Pi(d) \in 2^{F \times F} \Leftrightarrow \Pi(d) \subseteq F \times F$ collects the specific pairs
of impacts between which d has a weak preference

$$\Pi_d = \{(f, f') \in F \times F : d \text{ weakly prefers } f \text{ to } f'\}$$

What is a weak preference?

Weak preference: d accepts to cede f' for f (maybe the converse, too)

Definition: $f \preceq_d f' \Leftrightarrow (f, f') \in \Pi_d$

Economists often adopt the reverse notation $f \succeq_d f'$

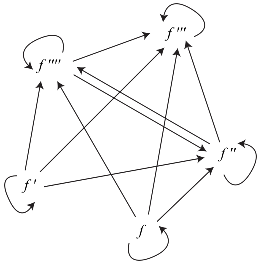
Representations of the preference function

In the finite case, a preference relation can be represented by

1. an incidence matrix
   (the rows and columns are impacts, the ones are preferences)

	f	f'	f''	f'''	f''''
f	1	0	1	1	1
f'	0	1	1	1	1
f''	0	0	1	1	1
f'''	0	0	0	1	0
f''''	0	0	1	1	1

2. a preference graph (the nodes are impacts, the arcs are preferences)



Derived relations

From the weak preference relation, one can derive

- indifference ($\mathrm{Ind}_\sqcap = \sqcap \cap \sqcap^{-1}$):

  Definition: $f \sim f' \Leftrightarrow f \preceq f'$ and $f' \preceq f$

  The decision-maker accepts the exchange in both directions

- strict preference ($\mathrm{Str}_\sqcap = \sqcap \setminus \sqcap^{-1}$):

  Definition: $f \prec f' \Leftrightarrow f \preceq f'$ and $f' \npreceq f$

  The decision-maker accepts the exchange only in the given direction

- incomparability ($\mathrm{Inc}_\sqcap = \bar{\sqcap} \cap \bar{\sqcap}^{-1}$)

  Definition: $f \bowtie f' \Leftrightarrow f \npreceq f'$ and $f' \npreceq f$

  The decision-maker rejects the exchange in both directions

We aim to describe all possible situations

Properties of binary relations

Some preference relations enjoy special properties

- reflexivity: $f \preceq f$ for all $f \in F$
- antisymmetry: if $f \preceq f'$ and $f' \preceq f$, then $f = f'$ for all $f, f' \in F$

  This forbids indifference
- completeness: if $f \not\preceq f'$, then $f' \preceq f$ for all $f, f' \in F$

  This forbids incomparability
- transitivity: if $f \preceq f'$ and $f' \preceq f''$, then $f \preceq f''$ for all $f, f', f'' \in F$

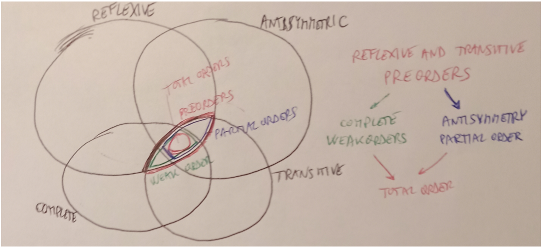
While building the model, we must investigate whether they hold or not because they could be

- useful to simplify the computation
- but unrealistic for the given problem

Main kinds of preference relation

Combining these properties, we can identify different kinds of preference

- a preorder enjoys reflexivity and transitivity
- a partial order enjoys reflexivity, transitivity and antisymmetry
  Example: subset inclusion
- a weak order enjoys reflexivity, transitivity and completeness
  Example: rankings in sport
- a total order enjoys reflexivity, transitivity, antisymmetry and completeness
  Example: number sizes



Conceptual problems on transitivity

Rational decision-makers are transitive, humans are not transitive

Homo oeconomicus non sum, num ratio a me aliena puto?

I am not an economist; am I not rational?

There are two main approaches to deal with this point

1. humans should be transitive, i.e., use thought experiments to show the evil consequences of giving up transitivity
2. let humans be nontransitive, i.e., appeal to practical and thought experiments and adopt weaker models of rationality

Gedankenexperiment 1: disproved incomparability

An incomparable pair can be made comparable by intermediate steps

Assume that $a \preceq b$ and $b \preceq c$, but $a \bowtie c$

The decision-maker

- does not accept to cede c and obtain a ($a \bowtie c$)

but

- accepts to cede c and obtain b ($b \preceq c$)
- accepts to cede b and obtain a ($a \preceq b$)

with the same final result (cede c and obtain a)

That does not make sense!

Gedankenexperiment 2: money pump

Assume that $a \prec b$ and $b \preceq c$, but $c \preceq a$
and that $a \sim b + \epsilon$ where ϵ is a small amount of money

The decision-maker

- accepts to cede c and obtain b ($b \preceq c$)
- accepts to cede b and ϵ and obtain a ($a \sim b + \epsilon$)
- accepts to cede a and obtain c ($c \preceq a$)
- and this can repeat an infinite number of times

paying ϵ every time to go back to the original situation (c)

That does not make sense!

Practical experiment 1: money pumps exist

The whole market of currencies is based on extracting money from cycles of exchanges between different currencies

The basic difference with the thought experiment is the role of time

- the exchanges take place in subsequent times
- the actual preferences vary in time

Consequently

- transitivity can be preserved using time-stamped impacts

$$(b, t_1) \preceq (c, t_1) \qquad (a, t_2) \sim (b + \epsilon, t_2) \qquad (c, t_3) \preceq (a, t_3)$$

- the number of impacts becomes huge ($F \to F \times T$) and the model becomes impractical

Gedankenexperiment 3: limited discrimination capacity

Given a sequence of coffee cups, with slowly increasing amounts of sugar, assume that the decision-maker

- prefers unsweetened coffee

$$0 \preceq \epsilon \preceq 2\epsilon \preceq \ldots \preceq M\epsilon$$

- is unable to recognise small differences in the quantity of sugar

$$k\epsilon \sim (k+1)\epsilon \quad \text{for all } k \in \mathbb{N}$$

Consequently, $(k+1)\epsilon \sim k\epsilon$ for all $k \in \mathbb{N}$, which implies

$$(k+1)\epsilon \preceq k\epsilon \text{ for all } k \in \mathbb{N} \quad \Rightarrow \quad M\epsilon \preceq (M-1)\epsilon \preceq \ldots \preceq \epsilon \preceq 0$$

The decision-maker prefers very sweet coffee (all impacts are indifferent)

That does not make sense!

Transitivity and limited discrimination imply no discrimination