MATHEURISTICS FOR COMBINATORIAL OPTIMIZATION PROBLEMS

Module 1- Lesson 3

Prof. Maurizio Bruglieri
Politecnico di Milano

OUTLINE- Lagrangean based matheuristics

- What is a relaxation of an ILP
- Relations among different relaxations of an ILP
- Methods to solve the Lagrangean dual problem
- Lagrangean based matheuristics for:
  - Generalized Assignment Problem
  - Single Source Capacitated Facility Location Problem
  - Min k-card cut

Mathematical Programming Relaxation

- Given a Mathematical Program (MP) *P*:

$$z = \min \ f(x)$$
$$\text{s.t. } x \in X$$

($X \subseteq \mathfrak{R}^n$ feasible solution set, $f: X \to \mathfrak{R}$ objective function)

we state that the MP, *R:*

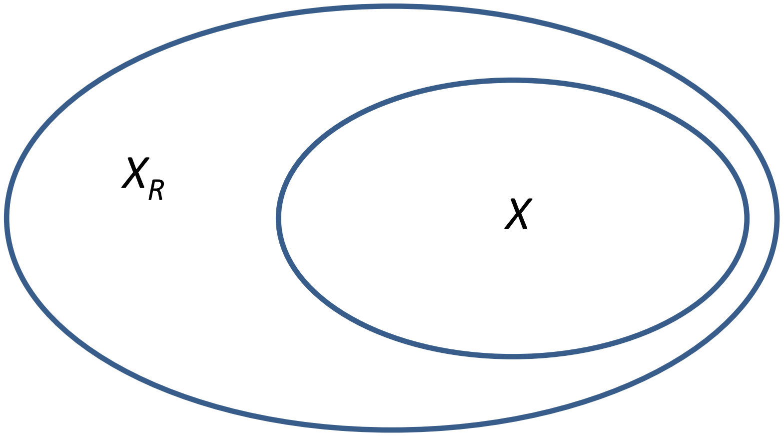
$$z_R = \min \ f_R(x)$$
$$\text{s.t. } x \in X_R$$

is a *relaxation* of *P* if and only if

i. $X \subseteq X_R$

ii. $f_R(x) \leq f(x) \ \forall x \in X$

(≥ for max problems)



Basic concepts on relaxations

- We look for relaxations *R* that are easier to be solved than *P*
- $z_R \leq z$ (e.g. z_R is useful to estimate the quality of a heuristic solution)
- If $f_R(x) = f(x)$ and the optimal solution of *R*, $x_R^* \in X \Rightarrow x_R^*$ is also optimal for *P*

Relaxation by elimination

$$Z(P) = \min \sum_{j=1}^{n} c_j x_j \quad (1)$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, \ldots, m), \quad (2)$$

$$\sum_{j=1}^{n} d_{kj} x_j = e_k \qquad (k = 1, \ldots, l), \quad (3)$$

$$x_j \in \{0, 1\} \qquad (j = 1, \ldots, n). \quad (4)$$

$$Z(E(P)) = \min \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, \ldots, m),$$

$$x_j \in \{0, 1\} \qquad (j = 1, \ldots, n),$$

Linear (or continuous) relaxation

$$Z(C(P)) = \min \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, \ldots, m),$$

$$\sum_{j=1}^{n} d_{kj} x_j = e_k \qquad (k = 1, \ldots, l),$$

$$0 \leq x_j \leq 1 \qquad (j = 1, \ldots, n),$$

- **Remark**: if its optimal solution is integer ⇒ it is also optimal for *P*

Surrogate relaxation

$$Z(S(P,\pi)) = \min \sum_{j=1}^{n} c_j x_j$$

$$\sum_{i=1}^{m} \pi_i \sum_{j=1}^{n} a_{ij} x_j \leq \sum_{i=1}^{m} \pi_i b_i$$

$$\sum_{j=1}^{n} d_{kj} x_j = e_k \qquad (k = 1, \ldots, l),$$

$$x_j \in \{0,1\} \qquad (j = 1, \ldots, n),$$

therefore $\sum_{i=1}^{m} \pi_i \sum_{j=1}^{n} a_{ij} x_j \leq \sum_{i=1}^{m} \pi_i b_i$ replaces the *m* constr. $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ with $\pi_i \geq 0$

- **Remark:** if its optimal solution is feasible for *P* ⇒ it is also optimal for *P*

Surrogate dual relaxation

$$Z\left(S(P,\pi^*)\right) = \max_{\pi\geq 0}\{Z(S(P,\pi))\}$$

Lagrangean relaxation

- **Idea**: remove the *m* constraints (2) and add them to the o.f. as penalty terms with coefficients $\lambda_i \geq 0$ (*Lagrangean multipliers*)

$$Z(L(P,\lambda)) = \min \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} \lambda_i (b_i - \sum_{j=1}^{n} a_{ij} x_j) = -\sum_{i=1}^{m} \lambda_i b_i + \min \sum_{j=1}^{n} \hat{c}_j x_j$$

$$\sum_{j=1}^{n} d_{kj} x_j = e_k \qquad (k = 1, \ldots, l),$$

$$x_j \in \{0,1\} \qquad (j = 1, \ldots, n)$$

with $\hat{c}_j = c_j + \sum_{i=1}^{m} \lambda_i a_{ij}, \forall j = 1, \ldots, n$

- **Remark**: if its optimal solution x^* is feasible for *P* it may be non optimal for *P* !
  It is optimal if it also satisfies:

$$\sum_{i=1}^{m} \lambda_i (b_i - \sum_{j=1}^{n} a_{ij} x_j^*) = 0$$

Lagrangean dual problem

$$Z(L(P,\lambda^*)) = \max_{\lambda \geq 0}\{Z(L(P,\lambda))\}$$

Relations among the relaxations

- The surrogate dual relaxation dominates the relaxation by elimination i.e., $Z\big(S(P,\pi^*)\big) \geq Z(E(P))$ (since $Z(E(P))$= $Z\big(S(P,0\,)\big)$)

- The Lagrangean dual relaxation dominates the relaxation by elimination i.e., $Z\big(L(P,\lambda^*)\big) \geq Z(E(P))$ (since $Z(E(P))$= $Z\big(L(P,0\,)\big)$)

Lagrangean dual vs Surrogate dual

- **Proposition 1:** The surrogate dual relaxation dominates the Lagrangean dual relaxation i.e., $Z\big(S(P,\pi^*)\big) \geq Z\big(L(P,\lambda^*)\big)$
  Proof: comparing the two relaxations the latter is a relaxation of the former:

$$
\begin{aligned}
Z(S_{(2)}(P,\mu)) = \ \min \ & \sum_{j=1}^{n} c_j x_j \\
& \sum_{i=1}^{m} \mu_i \sum_{j=1}^{n} a_{ij} x_j \leq \sum_{i=1}^{m} \mu_i b_i, \\
& \sum_{j=1}^{n} d_{kj} x_j = e_k \qquad (k = 1, \ldots, l), \\
& x_j \in \{0,1\} \qquad (j = 1, \ldots, n)
\end{aligned}
$$

$$
\begin{aligned}
Z(L_{(2)}(P,\mu)) = \ \min \ & \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} \mu_i b_i + \sum_{i=1}^{m} \mu_i \sum_{j=1}^{n} a_{ij} x_j \\
& \sum_{j=1}^{n} d_{kj} x_j = e_k \qquad (k = 1, \ldots, l), \\
& x_j \in \{0,1\} \qquad (j = 1, \ldots, n),
\end{aligned}
$$

Lagrangean dual vs continuous relaxation

- **Proposition 2:** The Lagrangean dual relaxation dominates the continuous relaxation i.e., $Z\big(L(P,\lambda^*)\big) \geq \mathrm{Z(C(P))}$

  Proof: it is based on the following Geoffrion's Lemma according to $Z\big(L(P,\lambda^*)\big)$ is equivalent to solve the original problem where the non-relaxed constraints are replaced by their convexification, i.e.,

  given $P = min\{cx: Ax = b, Ex = d, x \in \mathbb{Z}^n\} \Rightarrow$

  $Z\big(L(P,\lambda^*)\big) = min\{cx: Ax = b, x \in conv(X)\}$ where $X = \{x \in \mathbb{Z}^n: Ex = d\}$

  Since this feasible region is a subset of the continuous relaxation, $Z\big(L(P,\lambda^*)\big)$ provides a lower bound better than $\mathrm{Z(C(P))}$.

Lagrangean dual vs continuous relaxation

- **Geoffrion's Lemma:** Given $P = min\{cx: Ax = b, Ex = d, x \in \mathbb{Z}^n\} \Rightarrow$ $Z(L(P,\lambda^*)) = min\{cx: Ax = b, x \in conv(X)\}$ where $X = \{x \in \mathbb{Z}^n: Ex = d\}$ and $L(P,\lambda)$ is the Lagrangean relaxation of $Ax = b$ and $\lambda^*$the optimal solution of the Lagrangean dual problem

  Proof: Since $conv(X)$ is a polyhedron, $conv(X) = \{x \in \Re^n: \tilde{A}x \geq \tilde{b}\}$.

  Therefore $min\{cx: Ax = b, x \in conv(X)\}$= $min\{cx: Ax = b, \tilde{A}x \geq \tilde{b}\}$=

  by linear duality

$$= \max_{y,w}\{\ yb + w\tilde{b} : yA + w\tilde{A} = c, w \geq 0\ \}$$

$$= \max_y\{\ yb + \max_w\{\ w\tilde{b} : w\tilde{A} = c - yA, w \geq 0\ \}\ \}$$

  Then, applying again linear duality to the inner problem, one has:

$$= \max_y\{\ yb + \min_x\{\ (c - yA)x : \tilde{A}x \geq \tilde{b}\ \}\ \} = Z(L(P,\lambda^*))$$

Comparison between Lagrangean relaxations

- **Definition:** Given an ILP it has the *integrality property* if for any instance its continuous relaxation has integer optimal solution.

- **Proposition:** If the Lagrangean relaxation is applied to constraints having the *integrality property* ⇒ $Z\left(L(P,\lambda^*)\right) = \mathrm{Z(C(P))}$
  Proof: By Geoffrion's Lemma $Z\left(L(P,\lambda^*)\right) = min\{cx: Ax = b, x \in conv(X)\}$, but to the *integrality property* of X, $conv(X)$ reduces to $\{x \in \Re^n: Ex = d\}$

- Therefore when we apply Lagrangean relaxation it is important that the remaining problem is easy but not too much easy, i.e. $Ex = d$ must not to provide an exact formulation of $conv(X)$ otherwise we obtain the same (generally weak) bound of the continuous relaxation.

Example: comparison of Lagr. relax. for GAP

- **Definition:** Given *n* tasks and *m* agents each one with budget b_i and given the cost c_{ij} of performing task *j* with agent *i* and the consume a_{ij} of budget task *j* with agent *i* *Generalized Assignment Problem* (GAP) consists in assigning all the tasks in such a way that the total cost is minimized and the budget of each agent is not exceeded.

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \ \ \forall j = 1, \dots, n \quad (1)$$

$$\sum_{j=1}^{n} a_{ij} x_{ij} \leq b_i \ \ \forall i = 1, \dots, m \, (2)$$

$$x_{ij} \in \{0,1\} \ \ \forall i = 1, \dots, m, \forall j = 1, \dots, n \ \ (3)$$

Example: Lagr. relax. 1 for GAP

- We consider the LR of (1) with multipliers u_j:

$$L_1(u) = \min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} u_j \left(\sum_{i=1}^{m} x_{ij} - 1 \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + u_j) x_{ij} - \sum_{j=1}^{n} u_j$$

$$\sum_{j=1}^{n} a_{ij} x_{ij} \leq b_i \ \forall i = 1, \dots, m \qquad (2)$$

$$x_{ij} \in \{0,1\} \ \forall i = 1, \dots, m, \forall j = 1, \dots, n \qquad (3)$$

- This problem reduces to *m* knapsack problems that can be solved separately

Example: Lagr. relax. 2 for GAP

- We consider the LR of (2) with mulitpliers $v_i \geq 0$:

$$L_2(v) = \min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} v_i \left(\sum_{j=1}^{n} a_{ij} x_{ij} - b_i \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + v_i a_{ij}) x_{ij} - \sum_{i=1}^{m} v_i b_i$$

$$\sum_{i=1}^{m} x_{ij} = 1 \;\; \forall j = 1, \dots, n \quad (1)$$

$$x_{ij} \in \{0,1\} \;\; \forall i = 1, \dots, m, \forall j = 1, \dots, n \quad (3)$$

- This problem has the integrality property! ☹
- Indeed it can be solved considering for each j , $\min_i (c_{ij} + v_i a_{ij})$ and setting $x_{\bar{\imath}j} = 1$ where $\bar{\imath}$ realizes the minimum and the other variables $x_{ij} = 0$

How to solve Lagrangean dual problem

$$Z(L(P,\lambda^*)) = \max_{\lambda \geq 0}\{Z(L(P,\lambda))\}$$

- Iterative algorithms to compute λ^*
- Let λ^k the current Lagrangean multiplier vector and $x^*$the optimal solution of $L(P, \lambda^k)$
- If $x^*$is not optimal for *P* it is advantageous to set $\lambda_i^{k+1} = \max\{0, \lambda_i^k - \delta^k \left(b_i - \sum_{j=1}^{n} a_{ij}\ x_j^*\right)$ with step $\delta^k > 0$

Lagrangean dual problem

$$Z(L(P,\lambda^*)) = \max_{\lambda \geq 0}\{Z(L(P,\lambda))\}$$

$$\Updownarrow$$

$$Z(L(P,\lambda^*) = \max w$$

$$w \leq cx^h + \lambda(Ax^h - b), \forall h = 1, \ldots, q$$

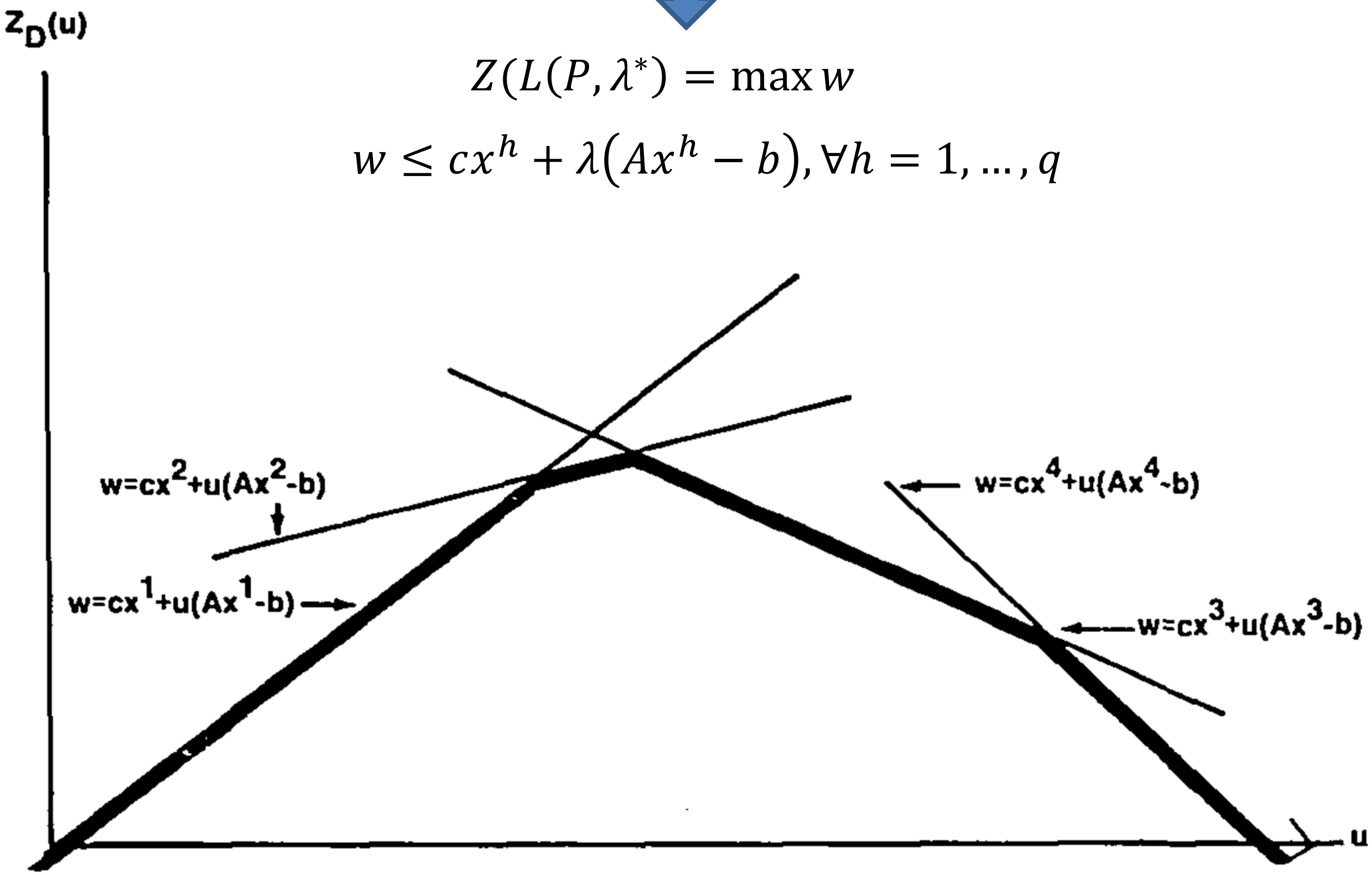


FIGURE 1. The Form of $Z_D(u)$.

Subgradient method

- **Definition**: $y \in \Re^m$ is a subgradient of $Z(\lambda)$ at $\bar{\lambda}$ if and only if
  $$Z(\lambda) \leq Z(\bar{\lambda}) + y(\lambda - \bar{\lambda})$$
- **Remark**: $Ax^h - b$ is a subgradient of $Z(L(P,\lambda))$ at λ being x^h the optimal solution of $L(P,\lambda)$
- The subgradient method is the extension of the gradient method to subdifferentiable functions: it leads again to the iterative formula
  $\lambda_i^{k+1} = \max\{0, \lambda_i^k - \delta^k \left(b_i - \sum_{j=1}^{n} a_{ij}\ x_j^*\right)$ with step $\delta^k > 0$
- Sufficient conditions to converge are: $\delta^k \rightarrow 0$ and $\sum_{k=1}^{\infty} \delta^k = \infty$
- Therefore $\delta^k = \frac{\alpha_k (Z^{UB} - Z(\lambda^k))}{\left\| Ax^k - b \right\|^2}$, $\alpha_k \in]0,2]$

A Lagrangean relaxation heuristic for SSCFLP

- **Single Source Capacitated Facility Location Problem (SSCFLP):**
  Given *n* customers and *m* possible facility locations, each customer *j* has an associated demand, q_j , that must be served by a single facility, each facility *i* has an overall capacity Q_i. The costs are composed of a cost c_{ij} for supplying the demand of a customer j from a facility established at location *i* and of a fixed cost, f_i , for opening a facility at location *i*.
  We want to decide which facilities opening and how to assign the customers to the facilities so that the overall cost is minimized.

$$\min \sum_{i \in I, j \in J} c_{ij} x_{ij} + \sum_{i \in I} f_i y_i \qquad (5.50)$$

$$s.t. \sum_{i \in I} x_{ij} = 1, \qquad j \in J \qquad (5.51)$$

$$\sum_{j \in J} q_j x_{ij} \leq Q_i y_i, \qquad i \in I \qquad (5.52)$$

$$x_{ij} \in \{0, 1\}, \qquad i \in I, j \in J \qquad (5.53)$$

$$y_i \in \{0, 1\}, \qquad i \in I \qquad (5.54)$$

A Lagrangean relaxation heuristic for SSCFLP

1. Relax in Lagrangean way the assignment constraints (5.51) obtaining:

$$z_{LR}(\boldsymbol{\lambda}) = \min \sum_{i \in I, j \in J} (c_{ij} - \lambda_j) x_{ij} + \sum_{i \in I} f_i y_i + \sum_{j \in J} \lambda_j$$

$$s.t. \sum_{j \in J} q_j x_{ij} \leq Q_i y_i, \qquad i \in I$$

$$x_{ij} \in \{0, 1\}, \qquad i \in I, j \in J$$

$$y_i \in \{0, 1\}, \qquad i \in I$$

2. Solve the LR by solving |I| knapsack problems separately:

$$z_{LR}^{i}(\lambda) = \min \sum_{j \in J} (c_{ij} - \lambda_j) x_{ij}$$

$$\sum_{j \in J} q_j x_{ij} \leq Q_i, \qquad x_{ij} \in \{0,1\}$$

3. For each $i \in I$, if $z_{LR}^{i}(\lambda) < -f_i \Rightarrow y_i = 1$, otherwise $y_i = 0$

A Lagrangean relaxation heuristic for SSCFLP

4. Check for unsatisfied constraints: the solution obtained may have cutomers assigned to multiple or no location. This can be detected by inspection. If the solution is feasible go to step 6, otherwise go to step 5.

5. Build a feasible solution: let $\bar{I}$ be the set of locations chosen in step 3. Solve the following GAP:

$$
\begin{aligned}
z_{GAP} = \min \sum_{i \in \bar{I}, j \in J} & c_{ij} x_{ij} \\
s.t. \sum_{i \in \bar{I}} x_{ij} &= 1, && j \in J \\
\sum_{j \in J} q_j x_{ij} &\leq Q_i, && i \in \bar{I} \\
x_{ij} &\in \{0, 1\}, && i \in \bar{I}, j \in J
\end{aligned}
$$

6. Update Lagrange multipliers by subgradient algorithm

A Lagr. relaxation heuristic for min k-card cut

Definition: Given an edge weighted undirected graph, the minimum k-cardinality cut problem (min k-cardinality cut) is the problem to find a cut such that the cut edge set C has cardinality k and the sum of the weights of the edges belonging to C is minimal

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} w(e) x_e \\ \text{subject to :} \quad & x \text{ be the incidence vector of a cut} \\ & \sum_{e \in E} x_e = k \end{aligned}$$

- The Lagrangean relaxation of the cardinality constraint provides:

$$\theta(\lambda) = \lambda k - maxcut(G, \lambda - w)$$

A Lagr. relaxation heuristic for min k-card cut

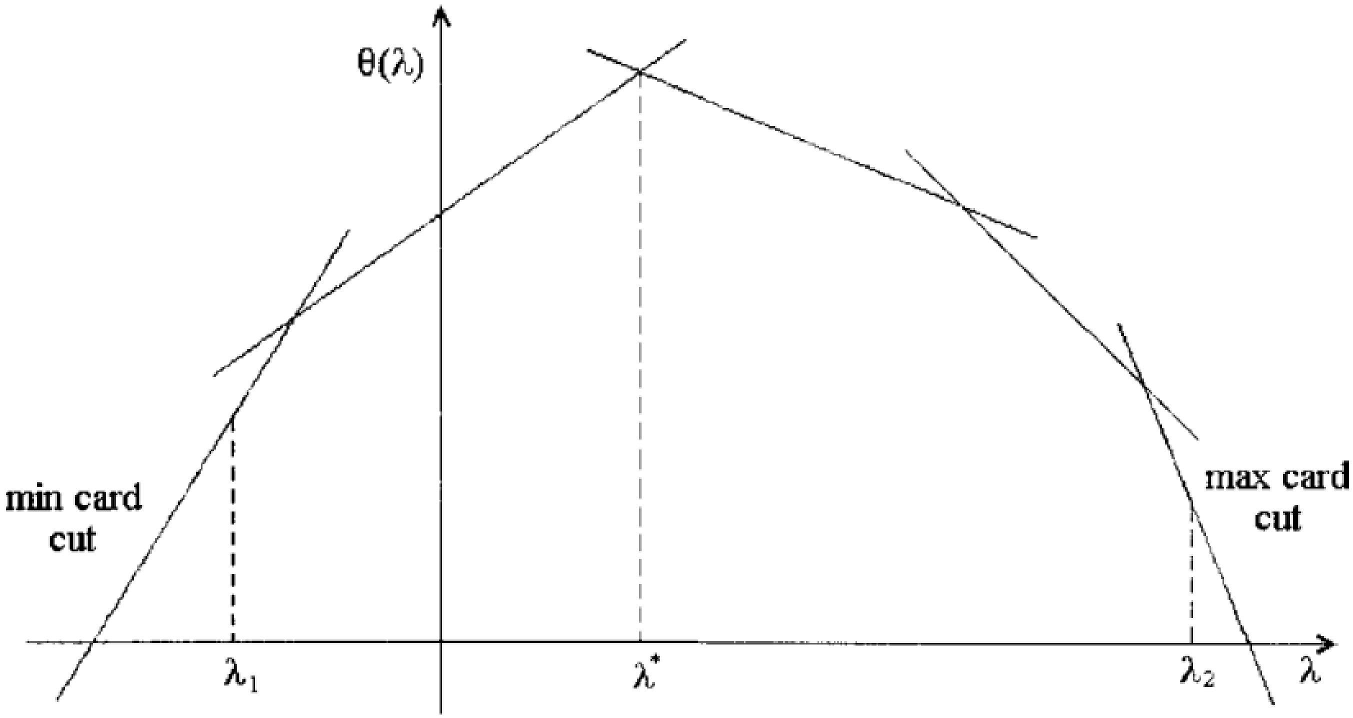


FIG. 7. Behavior of function $\theta(\lambda)$.

- Dicotomic search

A Lagr. relaxation heuristic for min k-card cut

Proposition 10.
Let

$$\lambda_1 := \max\left\{\frac{w(\widehat{E})}{|\widehat{E}|-k}, w(E^*) - w(\widehat{E})\right\}$$

and let

$$\lambda_2 := \min\left\{\frac{w(\widetilde{E})}{|\widetilde{E}|-k}, w(\widetilde{E}) - w(E^*)\right\}$$

where $\widehat{E}$ is the edge set of a min cardinality cut, E^* is the edge set of a min cut and $\widetilde{E}$ is the edge set of a max cardinality cut. For any $\lambda \leq \lambda_1$ the solution of maxcut$(G, \lambda - w)$ does not change from the min $|\widehat{E}|-$cardinality cut and for any $\lambda \geq \lambda_2$ the solution of maxcut$(G, \lambda - w)$ does not change from the min $|\widetilde{E}|-$cardinality cut.

A Lagr. relaxation heuristic for min k-card cut

- No approximation guarantee

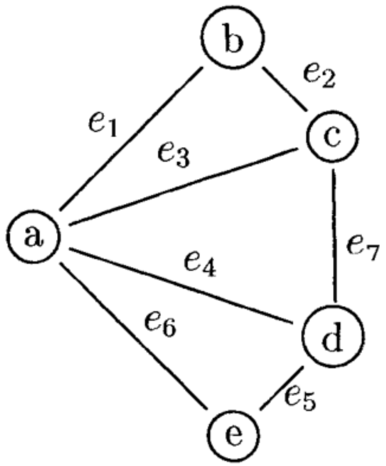


FIG. 9. Planar graph G having bad Lagrangian lower bound.

$$w(e) = \begin{cases} 1 & \text{for all } e \in E \setminus \{e_7\} \\ 1 + M & \text{for } e = e_7 \end{cases}$$

The optimal min 3-cardinality cut value is 3 + M.

A Lagr. relaxation heuristic for min k-card cut

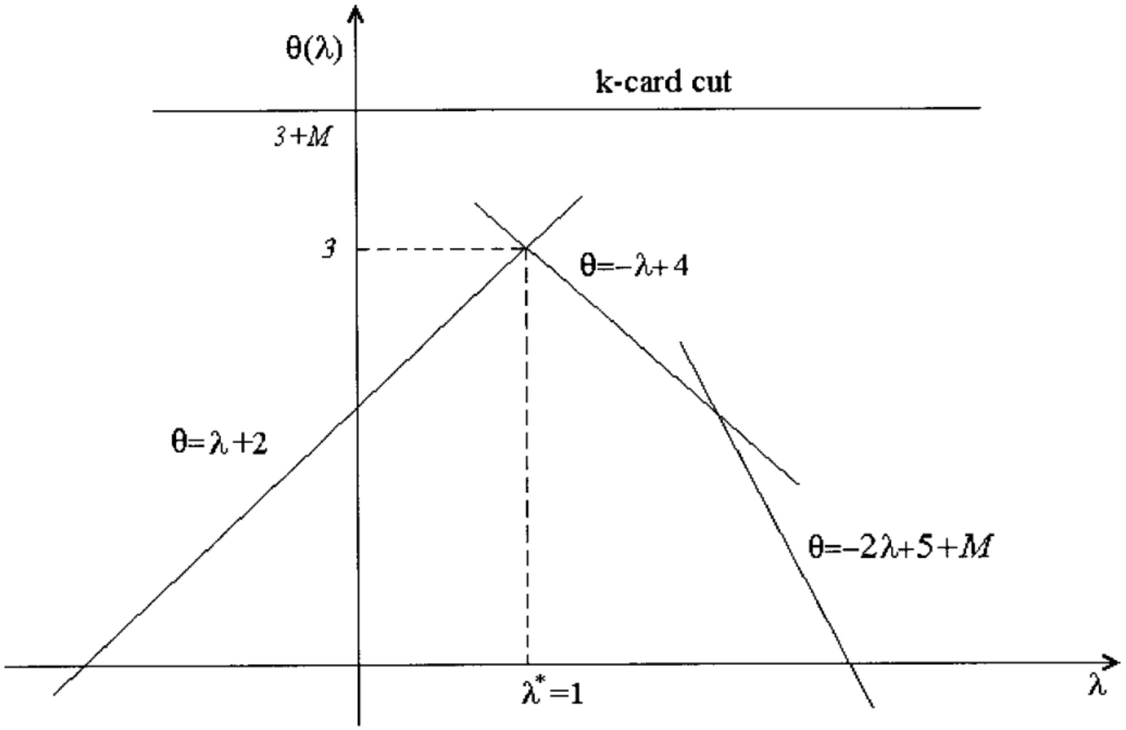


FIG. 10. Behavior of function $\theta(\lambda)$ for the graph drawn in Figure 9.

The Lagrangean relaxation is only 3.

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