Exercises on ILP formulations

1. Given the following set S of integer solutions:
S={(0,0,0,0),(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (0,1,0,1), (0,0,1,1)}and the two polyhedron:
P_1={ $x \in \Re^4 : 0 \le x \le 1, 83x_1 + 61x_2 + 49x_3 + 20x_4 \le 100$ }
P_2={ $x \in \Re^4 : 0 \le x \le 1, 4x_1 + 3x_2 + 2x_3 + x_4 \le 4$ }
a) verify that both P1 and P2 are formulations for S;
b) establish which of the two formulations is the best one.

Solution:
Since both polyhedra contain as integer solutions all and only the points in S, they are both formulations of S. The formulation P_2 is better than P_1 because $P_2 \subset P_1$ since multiplying by 25 both the members of the inequality characterizing P_2 we obtain the inequality $100x_1 + 75x_2 + 50x_3 + 25x_4 \le 100$ that has the same right hand side of P_1 but all the coefficients of the variables are smaller: this way we can see that e.g. point $(1, \frac{17}{61}, 0, 0)$ satisfies P_1 but not P_2.

2. Consider a transport problem with m possible sources (plants) and n destinations (customers). In many applications, the problem of determining which of the possible origins must work arises, since opening a source i generates a startup fixed cost F_i. Are also known costs c_{ij} to transport a single product from the source i to the destination j and the demand d_j of customer j. The aim is to determine the opening strategy of the plants and the transport plan with minimum total cost.
Let us introduce the variables $x_{ij} \ge 0$ to represent the quantity transported from origin i to destination j and the binary variables y_i such that:

$$y_i = \begin{cases} 1 \text{ if plant } i \text{ is active} \\ 0 \text{ otherwise} \end{cases}$$

The problem can be modeled as P_1:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} F_i y_i$$

$$\sum_{i=1}^{m} x_{ij} = d_j \quad \text{for } j=1,..,n \qquad (3.1)$$

$$\sum_{j=1}^{n} x_{ij} \le D y_i \quad \text{for } i=1,..,m \qquad (3.2)$$

$$x_{ij} \ge 0 \quad \text{for } i=1,..,m, \text{ for } j=1,..,n$$

$$y_i \in \{0,1\} \quad \text{for } i=1,..,m$$

with $D = \sum_{j=1}^{n} d_j$.

Another possible formulation is P_2 that differs from P_1 only in constraints (3.2) that are replaced with the following mn constraints:

$$x_{ij} \le d_j y_i \quad \text{for } j = 1, ..., n \ (3.3)$$

State and prove which of the two formulations is better.

Solution:

The formulation P_2 is better than the previous one, P_1 because if a vector (x, y) satisfies the constraints (3.3), adding both members of (3.3) for $j = 1, \ldots, n$, it is (x, y) satisfying also (3.2). Therefore $P_2 \subseteq P_1$. To prove that $P_2 \subseteq P_1$, it is necessary to show a point of P_1 that does not belong to P_2. Suppose for simplicity that m divides n, *i.e.* $n = km$ with $k \geq 2$ and integer. Then, a solution in which each source serves all the demand of k subsequent destinations, that is

$$x_{ij} = \begin{cases} d_j & for\ j = k(i-1)+1,\ldots,k(i-1)+k \\ 0 & otherwise \end{cases}, \text{ for } i = 1, \ldots, m$$

and $y_i = \frac{1}{D}\sum_{j=k(i-1)+1}^{k(i-1)+k} d_j$ for $i = 1, \ldots, m$, satisfies constraints (3.2) but not constraints (3.3): thus, such solution belongs to P_1 but not P_2.