Modeling, Analysis and Optimization of Networks (Part 2: design)

Lecture 3

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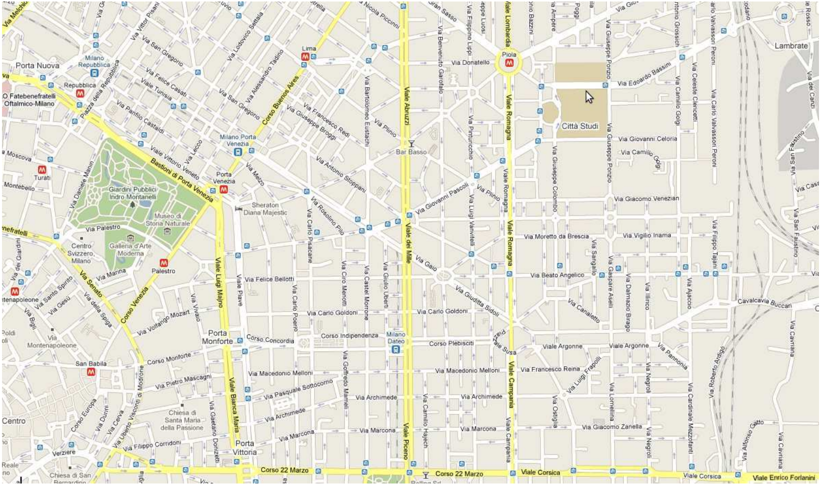
Connecting nodes on a network



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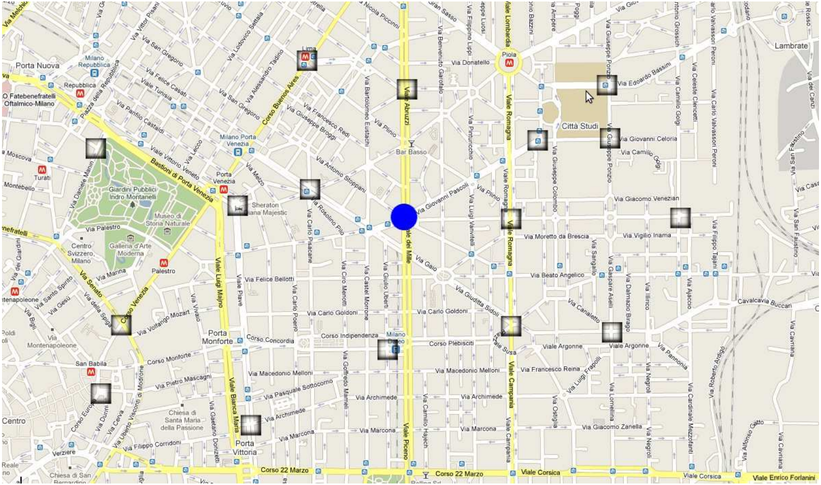

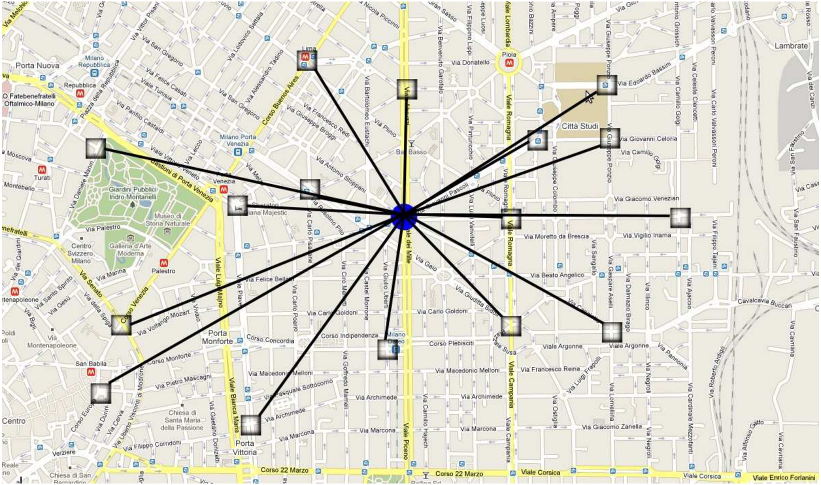
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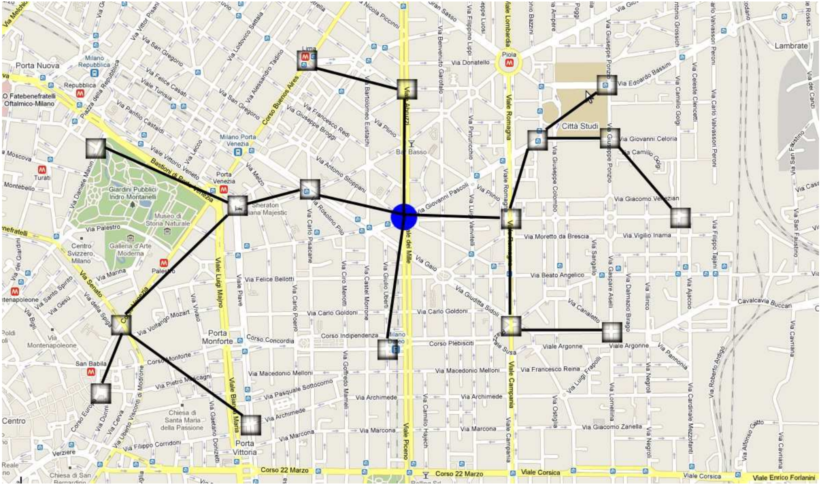

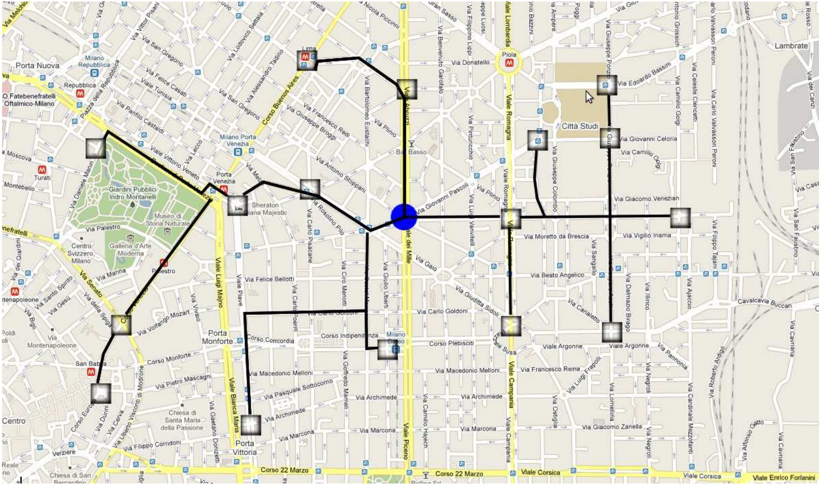
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The problem

Given

- a set of terminal nodes,
- a set of bridge nodes,
- a set of potential links connecting them,

I want to

- decide how to link nodes,
- in such a way that transmissions can be performed between each pair of terminal nodes,
- minimizing the network cost.

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- Each edge connecting vertices $i \in V$ and $j \in V$ has a cost c_{ij}.
- Find a **tree** in G of minimum total cost
- ...containg **all** terminals ($i \in T$) and **any subset** of the bridges ($i \in B$).

It is called the **Steiner Tree Problem (STP)** (Gauss, 1777-1855).

Solving the STP

Some considerations:

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- ... (it's not): MST is polynomially solvable, STP is NP-Hard.
- We'll see how to **approximate** it.

Approximation

What does it mean approximation?

Exact algorithms	a-priori guarantee of global optimality
Heuristics	no quality guarantee
Upper and lower bounds	a-posteriori quality guarantee
Approximation algorithms	a-priori quality guarantee

An α-approx algorithm **always** gives a solution of cost **at most** α times worse than the optimum.

A heuristic for the STP

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- STP asks to find a *minimum cost tree*
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- Good news: easy to compute.
- Bad news: such a tree might not be optimal (on the whiteboard, AA page 28).
- Question: what if an element of T exists having no neighbors in T?

The metric STP

The **metric** STP is a STP whose edge costs satisfy the **triangle inequality**: given three vertices $i, j, k \in V$

$$c_{ij} \leq c_{ik} + c_{kj}$$

Theorem: there is an approximation factor preserving reduction from the STP to the metric STP (proof on the whiteboard, AA page 27).

Approximating the metric STP

Metric STPs have a better structure:

Theorem: (for the metric STP), the cost of an MST on T is within 2-OPT (proof on the whiteboard, AA page 28).

Approximating the STP

A 2-approx algorithm for the STP is the following:

- given a STP instance on a graph G, build an (equivalent) instance of the metric STP on a graph G'
- find a MST on terminals in graph G'
- map edges of G' in this MST to edges in G

Steiner forests

Let us generalize the STP as follows: given

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- A set of **connection requests** between terminals: for each pair of terminals $s, t \in T$, coefficients $r_{st} = 1$ if s and t must be connected, $r_{st} = 0$ otherwise.

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- Each edge connecting vertices $i \in V$ and $j \in V$ has a cost c_{ij}.
- A set of **connection requests** between terminals: for each pair of terminals $s, t \in T$, coefficients $r_{st} = 1$ if s and t must be connected, $r_{st} = 0$ otherwise.
- Find a **forest** in G of minimum total cost
- ...containing **at least one path** connecting each pair of terminals s and t having $r_{st} = 1$.

It is called the **Steiner Forest Problem (SFP)**.

Mathematical programming models

Let us consider a *cut function*: for each $S \subseteq V$

$$f(S) = \begin{cases} 1 & \text{if } S \text{ contains } s \text{ and } V \setminus S \text{ contains } t \text{ such that } r_{st} = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let us consider *crossing sets*: for each $S \subseteq V$

$$\delta(S) = \text{set of edges crossing the cut } (S, V \setminus S)$$

Mathematical programming models

Primal:

$$\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in E} c_{ij} x_{ij} & \\
\text{subject to} \quad & \sum_{(i,j)\in\delta(S)} x_{ij} \geq f(S) & \forall S \subseteq V \\
& x_{ij} \in \{0,1\} & \forall (i,j) \in E
\end{aligned}$$

Mathematical programming models

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Dual:

$$\begin{aligned} \text{maximize} \quad & \sum_{S\subseteq V} f(S) y_S & \\ \text{subject to} \quad & \sum_{S:(i,j)\in\delta(S)} y_S \leq c_{ij} & \forall (i,j) \in E \\ & y_S \geq 0 & \forall S \subseteq V \end{aligned}$$

Figurative terminology

- set S *has been raised* if $y_S > 0$ (remark: it is never convenient to raise sets S having $f(S) = 0$)
- edge (i, j) *feels* dual y_S if $(i, j) \in \delta(S)$ and $y_S > 0$
- edge (i, j) *is tight* (resp overtight) if the sum of duals it feels equals (resp exceeds) its cost

Optimality conditions

Primal (slackness) conditions: for each $(i,j) \in E$,

$$x_{ij} \neq 0 \rightarrow \sum_{S:(i,j)\in\delta(S)} y_S = c_{ij}$$

Dual (slackness) conditions: for each $S \subseteq V$,

$$y_S \neq 0 \rightarrow \sum_{(i,j)\in\delta(S)} x_{ij} = 1$$

Dual (relaxed slackness) conditions: for each $S \subseteq V$,

$$y_S \neq 0 \rightarrow \sum_{(i,j)\in\delta(S)} x_{ij} \leq 2 \cdot f(S) \text{ "on the average"}$$

(every raised cut has degree at most 2).

Primal-dual algorithm

Idea:

- start with a super-optimal (infeasible) primal and a sub-optimal (feasible) dual
- iteratively improve the feasibility of the primal and the optimality of the dual, until a feasible primal is obtained
- x_{ij} vars indicate which cuts need to be raised
- y_S vars indicate which edges need to be picked

Invariant: the set of vars x_{ij} always identifies a *forest*.

Primal-dual algorithm

A key step: **unsatisfied** and **active** sets. Given a primal solution, a set S is **unsatisfied** if

- has $f(S) = 1$
- there is no picked edge crossing the cut $(S, V \setminus S)$;

a set S is **active** if

- it is unsatisfied
- it does not contain unsatisfied sets (i.e. it is minimal wrt inclusion)

Lemma: A set S is active iff it is a connected component in the currently picked forest (and $f(S) = 1$).
(proof on the whiteboard, AA page 200).

Primal-dual algorithm for SFP

Primal-dual algorithm for SFP:

- (init) $x_{ij} := 0$; $y_S := 0$
- (augmentation) while there exists an unsatisfied set S do
  find active sets (by listing connected components)
  simultaneously raise y_S for each active set S
  until some edge (ij) becomes tight
  set $x_{ij} := 1$ for each tight edge
- (pruning) for each (i, j) such that $x_{ij} = 1$, set $x_{ij} := 0$ if the primal solution remains feasible

(example on the dashboard, AA pages 202-204).

Analysis

Theorem: Primal-dual algorithm for the SFP achieves an approximation guarantee of 2.
(proof on the whiteboard, AA pages 204-206)

Tightness of the analysis

Are the analyses tight?

- try to find a STP (or SFP) instance in which our algorithms reach the worst case guarantee ...

example: page 30 AA.

Further remarks

Some final observations. Both STP and MST are special cases of SFP:

- when run on a STP instance, the primal-dual algorithm *builds a Spanning Tree* on set T
- $\rightarrow$ the MST algorithm for STP is a special case of the primal-dual;
- when run on a MST instance (i.e. $T = V$), the primal-dual algorithm is essentially Kruskal's algorithm.